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Numeri Complessi

Motivazione: non esistono radici reali di $x^2+1=0$

ossia $x^2+1=0$ è irriducibile su $\mathbb{R}$, ossia $\nexists\, x\in\mathbb{R} : x^2=-1$

$i : i^2+1=0$ i = unità immaginaria

Def Un numero complesso è una coppia ordinata $(x,y)\in\mathbb{R}\times\mathbb{R}$

$z\equiv(x,y)$, $z=x+iy$ con $i^2=-1$

$\mathbb{C}$ è l'insieme dei numeri complessi

$$\mathbb{C}:=\{x+iy \,,\, x,y\in\mathbb{R}\}\qquad \begin{array}{l} z_1=x_1+iy_1\\ z_2=x_2+iy_2\end{array}$$

Addizione
$$z_1+z_2=(x_1+iy_1)+(x_2+iy_2)$$
$$=(x_1+x_2+i(y_1+y_2))$$

Prodotto
$$z_1\cdot z_2=(x_1+iy_1)\cdot(x_2+iy_2)$$
$$=x_1x_2+ix_1y_2+iy_1x_2+i^2y_1y_2$$
$$=(x_1x_2-y_1y_2)+i(x_1y_2+y_1x_2)$$

$$z=\mathrm{Re}(z)+i\,\mathrm{Im}(z)=x+iy$$

$x\equiv\mathrm{Re}(z)\in\mathbb{R}$, $y\equiv\mathrm{Im}(z)\in\mathbb{R}$

parte reale di z — parte immaginaria di z

Proposizione $\mathbb{C} := (\mathbb{C}, +, \cdot, 0, -z, 1, z^{-1})$ è un campo commutativo (abeliano).

ad esempio $z \mapsto -z = -x + i(-y) \equiv (-x, -y)$
passaggio all'opposto

$\frac{z_1}{z_2}$ è possibile se $z_2 \neq 0 = (0,0) = 0 + i \cdot 0$

$$\frac{1}{z} \left(= z^{-1}\right) = \frac{1}{x+iy} = \frac{(x-iy)}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}$$

$$= \frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2}, \text{ ossia}$$

$$z = x + iy \longmapsto z^{-1} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} \quad \text{reciproco}$$

$$\mathrm{Re}(z^{-1}) = \frac{x}{x^2+y^2}, \quad \mathrm{Im}(z^{-1}) = \frac{-y}{x^2+y^2}.$$

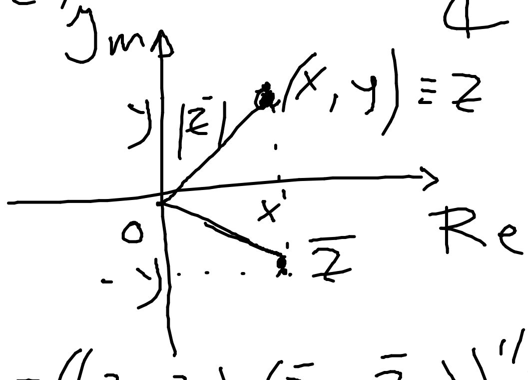
$\bar{z}$ = complesso coniugato di z
$= x - iy$ si ha:

$$z^{-1} = \frac{\bar{z}}{|z|^2} \text{ con } |z|^2 = z\bar{z} = \bar{z}z = x^2 + y^2$$

Modulo ($\equiv$ norma, valore assoluto di z)

$$|z| := (z\bar{z})^{1/2} = (x^2+y^2)^{1/2}$$

distanza dall'origine

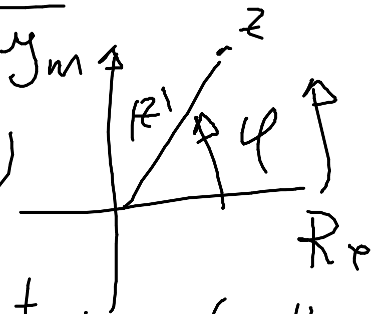


Distanza (euclidea) su $\mathbb{C}$:

$$|z_1 - z_2| := \left((x_1-x_2)^2 + (y_1-y_2)^2\right)^{1/2} = \left((z_1-z_2)\cdot(\bar{z}_1 - \bar{z}_2)\right)^{1/2}$$

$z = x + iy$ presentazione cartesiana

φ_z (anomalia) argomento di z "$\arg(z)$"



$\mathbb{C}$

$$\begin{cases} \operatorname{Re} z = |z| \cos(\arg z) \\ \operatorname{Im} z = |z| \sin(\arg(z)) \end{cases}$$

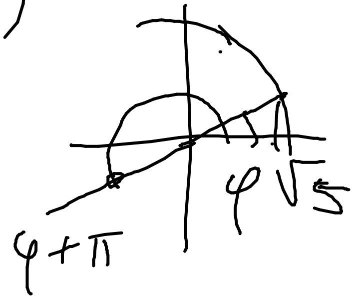
$(|z|, \arg(z))$ presentazione trigonometrica (polare)

- $\arg(z)$ non è definito se $z = 0$
- $\arg(z)$ è definito a meno di $K2\pi$, $K \in \mathbb{Z}$

$$z = x + iy = |z|(\cos\varphi + i \sin\varphi)$$
$$= |z|(\cos(\varphi + K2\pi) + i \sin(\varphi + K2\pi)), K \in \mathbb{Z}$$

Esempi $z = 2 + i = (x, y) = (2, 1)$

$$|z| = \sqrt{4+1} = \sqrt{5}$$
$$\varphi = \operatorname{arctg}\left(\frac{y}{x}\right) = \operatorname{arctg}\left(\frac{1}{2}\right)$$



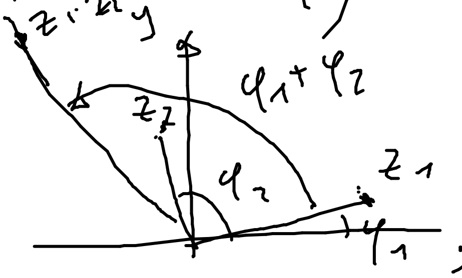
Formula di De Moivre

Proposizione $z_1, z_2 \in \mathbb{C}$, vale $|z_1 z_2| = |z_1||z_2|$

e $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2) + K2\pi$, $K \in \mathbb{Z}$.

dim
$$z_1 \cdot z_2 = |z_1|(\cos\varphi_1 + i \sin\varphi_1)|z_2|(\cos\varphi_2 + i \sin\varphi_2)$$
$$= |z_1| \cdot |z_2| (\cos\varphi_1 \cdot \cos\varphi_2 - \sin\varphi_1 \sin\varphi_2 +$$
$$i(\cos\varphi_1 \sin\varphi_2 + \sin\varphi_1 \cos\varphi_2))$$

(form. addizione) $= |z_1||z_2| (\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2))$ $\square$



$\mathbb{C}$

Se $\frac{z_1}{z_2} = z$, $z_2 \neq 0$ $\iff$ $z_1 = z \cdot z_2$ $\begin{cases} |z||z_2| = |z_1| \\ \varphi + \varphi_2 = \varphi_1 \end{cases}$

<u>Corollario</u> $z_1, z_2 \in \mathbb{C}$, $z_2 \neq 0$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2k\pi \quad k \in \mathbb{Z}$$

Iterando la proposizione:

$\forall n \in \mathbb{N}$ $\quad |z^n| = |z|^n$, $\arg(z^n) = n \arg(z) + 2k\pi$

$(n \in \mathbb{Z}?)$ $\quad z^n = |z|^n (\cos(n\varphi) + i \sin(n\varphi))$

$$z = |z| (\cos(\varphi) + i \sin(\varphi))$$

<u>in particolare</u> se $|z| = 1$

$$\boxed{z^n = (\cos\varphi + i \sin\varphi)^n = \cos(n\varphi) + i \sin(n\varphi)}$$

<u>Formula di De Moivre</u>

$a > 0$ reale in particolare $a = e$.

<u>Notazione</u>: $e^{i\varphi} := \cos\varphi + i \sin\varphi$, $\varphi \in \mathbb{R}$

$z_1, z_2 \in \mathbb{C}$ di modulo $= 1$

$$z_1 \cdot z_2 = (\cos\varphi_1 + i \sin\varphi_1)(\cos\varphi_2 + i \sin\varphi_2)$$
$$= e^{i\varphi_1} \cdot e^{i\varphi_2} = e^{i(\varphi_1 + \varphi_2)}$$
$$= \cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2)$$

$\boxed{\varphi \text{ fase}}$

e De Moivre $\quad (e^{i\varphi})^n = e^{in\varphi}$

la notazione esponenziale trasforma prodotti $(z_1 \cdot z_2, z^n, \ldots)$ in somme $(\varphi_1 + \varphi_2, n \cdot \varphi, \ldots)$

In particolare

$$e^{2k\pi i} = 1 \; ; \; e^{i(\varphi+2k\pi)} = e^{i\varphi} \; , \; k \in \mathbb{Z}$$

$$|e^{i\varphi}| = 1 \; ; \; z = |z| e^{i\varphi} = |z| e^{i(\varphi+2k\pi)} \; , \; k \in \mathbb{Z}$$

$$\varphi = \frac{\pi}{2} \left(/ \, 2k \, z \right) \qquad e^{i\frac{\pi}{2}} = i \; ; \; e^{i\pi} = -1 \; ; \; e^{-i\frac{3}{2}\pi} = -i$$

$z = |z| e^{i\varphi}$ presentazione esponenziale o polare

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2} \; , \; \sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2}$$

Esempio $z = \dfrac{i+5}{3+2i}$ trovare la forma cartesiana trigonometrica ed esponenziale di z.

$$\bar{z} = \frac{-i+5}{3+i} \qquad z = x+iy \qquad , y, x \in \mathbb{R} .$$

$$z = \frac{i+5}{3+2i} \cdot \frac{3-2i}{3-2i} = \frac{(15+2) + i(-7)}{9+4} = \frac{17}{13} + i\left(-\frac{7}{13}\right)$$

$$|z| = \left(\left(\frac{17}{13}\right)^2 + \left(\frac{7}{13}\right)^2 \right)^{1/2} = \frac{1}{13} \left(17^2 + 7^2 \right)^{1/2}$$

$$\varphi = \operatorname{arctg}\left(-\frac{7}{\cancel{13}} \frac{\cancel{13}}{17} \right) = -\operatorname{arctg}\left(\frac{7}{17}\right)$$

$\leadsto$ forma trigonometrica ed esponenziale

Radici Complesse

Proposizione $w \in \mathbb{C}$, $n = 2, 3, \ldots$

i) Se $w = 0$, $\exists!$ soluzione $z = 0$ di $z^n = w$;

ii) Se $w \neq 0$, $z^n = w$ ammette n (differenti) soluzioni complesse 6

$$z_K = |w|^{\frac{1}{n}} e^{i(\arg(w) + 2K\pi)/n}$$

$$K = 0, 1, 2, \dots, n-1$$

Dim i) $w = 0 \Leftrightarrow z^n = 0 \Leftrightarrow |z^n| = 0 \Leftrightarrow |z|^n = 0$

$\Leftrightarrow z = 0$

ii) $w \neq 0$

$$z^n = w \Leftrightarrow \begin{cases} |z^n| = |w| \\ \arg(z^n) = \arg(w) + 2K\pi, \quad K \in \mathbb{Z} \end{cases}$$

$\Leftrightarrow n \arg(z) = \arg(w) + 2K\pi$

(2°) $\quad \arg(z) = (\arg(w) + 2K\pi)/n \quad , K \in \mathbb{Z}$

In particolare $K = 0, 1, \dots, n-1$

oppure $K_0 \in \mathbb{Z} \rightsquigarrow K_0, K_0+1, K_0+2, \dots K_0+(n-1)$

$\square$

Esempi. $z^4 - 6 = 0 \Leftrightarrow z^4 = 6 \ (\neq 0)$

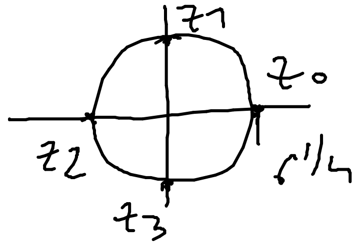
$$z = \underbrace{|z|^4}_{\rho} e^{i\varphi 4} = 6\, e^{i(0 + 2K\pi)} \qquad K \in \mathbb{Z}$$

radici sono 4 : $\varphi_K = \dfrac{\cancel{2} K\pi}{\cancel{4}_2} = \dfrac{K\pi}{2}$

φ_K

$K = 0, 1, 2, 3 \qquad \varphi_0 = 0, \ \varphi_1 = \dfrac{\pi}{2}, \ \varphi_2 = \pi, \ \varphi_3 = \dfrac{3}{2}\pi$

$z_0 = 6^{1/4}$, $z_1 = 6^{1/4} \underbrace{e^{i\frac{\pi}{2}}}_{i} = i6^{1/4}$, $z_2 = 6^{1/4} \underbrace{e^{i\pi}}_{-1} = -6^{1/4}$, $z_3 = 6^{1/4} \underbrace{e^{i\frac{3}{2}\pi}}_{-i} = -i6^{1/4}$



Esercizi

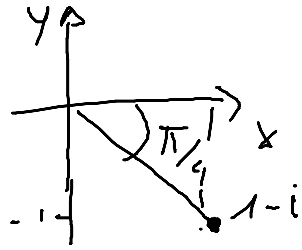
1.1 $(z^2+i)^2 = 1$

$z^2+i = w \in \mathbb{C}$

$w^2 = 1 \iff w = \pm 1$

$\boxed{\text{per } w=1}$ $\quad z^2+i = 1 \iff z^2 = 1-i$

$1-i = \rho = \sqrt{2} = (x^2+y^2)^{1/2}$



$\rho \geq 0$, φ (fase) $= -\frac{\pi}{4}$

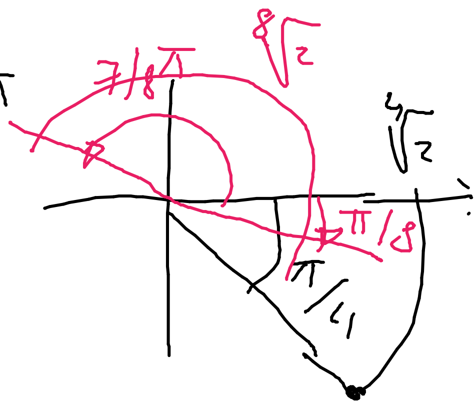
$1-i = \sqrt{2}\, e^{-i\frac{\pi}{4}}$

$z^2 = 2^{\frac{1}{4}}\, e^{-i\pi/4}$

$z_0 = 2^{\frac{1}{8}}\, e^{-i\frac{\pi}{4}/2} = 2^{\frac{1}{8}}\, e^{-i\frac{\pi}{8}}$, $\varphi_0 = -\frac{\pi}{8}$

$|z_1| = 2^{1/8}$, $\varphi_1 = \left(-\frac{\pi}{4} + 2\pi\right)/2 = \frac{7}{8}\pi$

$z_1 = 2^{1/8}\, e^{i\frac{7}{8}\pi}$



$\boxed{w=-1}$ $\quad z^2+i = -1$

$z^2 = -(1+i)$ altre due soluzioni

$\boxed{\text{da finire}}$

1.4 $z^6 + iz^3 = 0 \qquad w = z^3 \qquad w^2 + iw = 0$

$w(w+i) = 0$

quindi $w = 0 \iff z = 0$ (tripla)

$w+i = 0 \iff z^3 + i = 0 \rightarrow$ 3 radici (calcolare)

oppure $z^3(z^3+i)=0$ — $z=0$ ...
$z^3+i=0 \rightsquigarrow$ tre radici.

1.9

$$(z^3+|z|)(z^2+z+i)=0 \qquad A(z)\cdot B(z)=0$$

$$A(z)=0 \quad =\left(|z|^3e^{i3\varphi}+|z|\right)=0$$

$z=\rho e^{i\varphi}$

$0\le\rho=|z|$ cioè $\rho\left(\rho^2e^{i3\varphi}+1\right)=0$

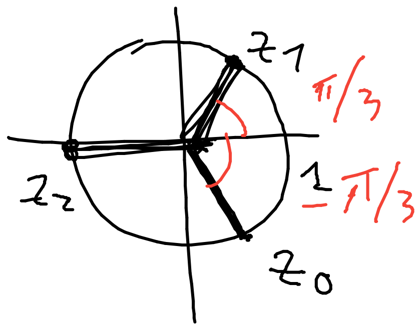
Soluzioni $\rho=0 \iff \boxed{z=0}$

$$\rho^2e^{i3\varphi}=e^{-i\pi} \iff \begin{cases}\rho^2=1 \iff \boxed{\rho=1}\\ 3\varphi=-\pi+2k\pi,\ k\in\mathbb{Z}\end{cases}$$

$$\varphi=\varphi_0=-\pi/3 \qquad z_0=e^{-i\pi/3}$$

$$\varphi_1=\frac{-\pi+2\pi}{3}=\pi/3 \qquad z_1=e^{i\pi/3}$$

$$\varphi_2=\frac{-\pi+4\pi}{3}=\pi \qquad z_2=e^{i\pi}=-1$$



$B(z):\ z^2+z+i=0$

usando formula per $P_2(z)$

$\rightsquigarrow$ 2 soluzioni $\quad z_{1,2}=\dfrac{-1\pm(1-4i)^{1/2}}{2}$ (da completare)

oppure

$z=x+iy$ (proviamo)

$$(x+iy)^2+(x+iy)+i=0=0+i\cdot 0$$

$$\begin{cases}\mathrm{Re}\ \ x^2-y^2+x=0\\ \mathrm{Im}\ \ 2xy+y+1=0\end{cases}$$

(non sembra possibile)

2.3 $\left(z+|z|-1\right)\left(z^2+\frac{8}{z^2+4}\right)=0$ $\boxed{\text{definita se } z^2+4\neq 0 \iff z\neq \pm 2i}$

$A(z,\bar z)\cdot B(z,\bar z)=0$

oss z^2 non è positivo, in generale

$A: x+iy+(x^2+y^2)^{1/2}-1=0 \iff \begin{cases} x+(x^2+y^2)^{1/2}=1 \\ y=0 \end{cases}$

quindi $y=0$ e $z=x$ è reale

$x+(x^2+y^2)^{1/2}=1 \iff x+|x|=1$

$\iff 2x=1 \iff x=\frac{1}{2}$ se $x>0$

$x-x=0\neq 1$ se $x<0$ non ha soluzioni

$A \rightsquigarrow \boxed{z=1}$ soluzione

$B(z)= z^2+\frac{8}{z^2+4}=0 \iff (z^2+4)z^2+8=0$

per $z\neq \pm 2i$ come biquadratiche

(da terminare)