Exe 1: Esercizi sui numeri complessi

CALCOLARE LE SEGUENTI ESPRESSIONI:

1. $1 + i + (i)^2 + (i)^3 + (i)^4 + \dots + i^{50}$

2. $(1+i)^{10}$.

3. $\dfrac{(i-\sqrt{3})^{69}}{(1+i)^{139}}$

4. $\left((\sqrt{3}-1) + i(\sqrt{3}+1)\right)^{12}$

TROVARE (IN FORMA CARTESIANA) TUTTE LE RADICI n-ESIME DI z NEI SEGUENTI CASI:

7. $z = i \qquad n = 2, 3, 4$

10. $z = \dfrac{(5+12i)^4}{(7-24i)^2} \qquad n = 8$

CALCOLARE

12. LA SOMMA E IL PRODOTTO DI TUTTE LE RADICI 16^e DI $1+i$

13. LA SOMMA DEI QUADRATI DI TUTTE LE RADICI OTTAVE DI $-8-8\sqrt{3}i$ AVENTI PARTE IMMAGINARIA NEGATIVA.

RISPONDERE AI SEGUENTI QUESITI:

17. DATI $z_1 = -\sqrt{3} + i$ E $z_2 = \sqrt{2} + i\sqrt{2}$,
   - a) TROVARE IN FORMA CARTESIANA z_1^{96} E z_2^{96};
   - b) PRESI $n \in \mathbb{N} - \{0\}$ E $z \in \mathbb{C}$ TALI CHE SIA z_1 CHE z_2 SIANO RADICI n-ESIME DI z, DIRE QUAL È IL MINIMO VALORE POSSIBILE PER $|z|$.

18. DATI $z_1 = \sqrt{2+\sqrt{3}} + i\sqrt{2-\sqrt{3}}$ E $z_2 = \sqrt{2-\sqrt{2}} - i\sqrt{2+\sqrt{2}}$,
   - a) TROVARE z_1^{2022} E, IN PARTICOLARE, $\left|\operatorname{Im} z_1^{2022}\right|$.
   - b) DETERMINARE L'INSIEME $I = \{ n \in \mathbb{Z} \mid z_1^n = z_2^n \}$

19 DOPO AVER RISOLTO L'EQUAZIONE $z^{18}=z^{10}$ TROVARE TUTTI I NUMERI COMPLESSI PER I QUALI L'INSIEME DELLE RADICI DICIOTTESIME NON È DISGIUNTO DA QUELLO DELLE RADICI DECIME.

RISOLVERE IN $\mathbb{C}$ LE SEGUENTI EQUAZIONI:

20 $z^4=z^2+2$

21 $z^4=|z|^2+2$

22 $z^{10}\cdot\bar{z}^8=512i$

RISPONDERE AI SEGUENTI QUESITI:

34 DEL NUMERO COMPLESSO z SAPPIAMO SOLO CHE $z^{24}=17+i$.
QUANTI DIVERSI VALORI PUÒ ASSUMERE z^{15}?

37 DEL POLINOMIO $p(z)$ SAPPIAMO CHE:

a) I COEFFICIENTI SONO TUTTI INTERI,

b) È DIVISIBILE PER $q(z)=z^2+z+1$,

c) SE z_0 È SOLUZIONE DI $p(z)=0$ ALLORA ANCHE iz_0 LO È.

QUAL È IL MINIMO GRADO CHE PUÒ AVERE $p(z)$?

SOLUZIONI

1

$$1 + i + (i)^2 + (i)^3 + (i)^4 + \dots + i^{50} = 0 + i^{48} + i^{49} + i^{50} = \cancel{1} + i - \cancel{1} = i$$

$$\underbrace{1 + i - 1 - i}_{0} \qquad \underbrace{(i)^4 + (i)^5 + (i)^6 + (i)^7}_{1 \;\; i \;\; + i^2 + i^3}$$

$$\overbrace{1 + i + \dots + i^{47}}^{0}$$

$$\boxed{1 - x^3} = (1 - x)(1 + x + x^2)$$

II MODO

$$\frac{(1 + i + i^2 + i^3 + \dots + i^{50})(1 - i)}{1 - i} = \frac{1 - i^{51}}{1 - i} = \boxed{\frac{1 - i^3}{1 - i}} =$$

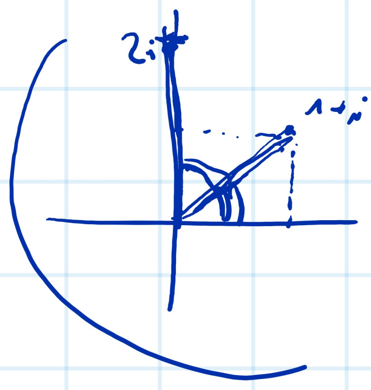
$$= \frac{\cancel{(1 - i)}(1 + i + i^2)}{\cancel{1 - i}} = 1 + i - 1 = i$$

$$\frac{1 - i^3}{1 - i} = \frac{1 + i}{1 - i} = \frac{(1 + i)^2}{(1 - i)(1 + i)} = \frac{\cancel{1} - \cancel{1} + 2i}{2} = \frac{2i}{2} = i$$

2

$$(1 + i)^{10} = \left(\underbrace{(1 + i)^2}\right)^5 = (2i)^5 = 32i$$

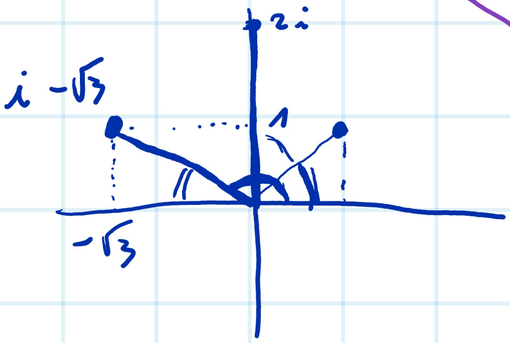
$$\cancel{1^2} + \cancel{i^2} + 2i = 2i$$

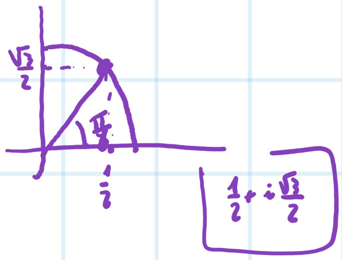


3

$$\frac{(i - \sqrt{3})^{69}}{(1 + i)^{139}} =$$

$$= \left(\frac{i-\sqrt{3}}{\underbrace{(1+i)^2}_{2i}}\right)^{69}\cdot\frac{1}{1+i} =$$

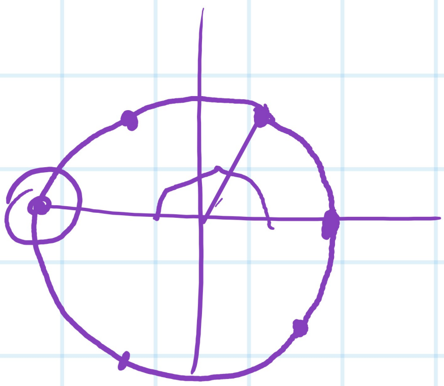




$$\frac{i-\sqrt{3}}{2i} = \frac{1}{2} - \frac{\sqrt{3}}{2i} = \frac{1}{2} - \frac{\sqrt{3}}{2i^2}\cdot i =$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$= \left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{69}\cdot\frac{1}{1+i} =$$



$$= \Bigg(\underbrace{\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{6}}_{1}\Bigg)^{11}\cdot\underbrace{\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{3}}_{-1}\cdot\frac{1}{1+i}$$

$$= 1\cdot(-1)\cdot\frac{1\cdot(1-i)}{\underbrace{(1+i)(1-i)}_{2}} = \boxed{-\frac{1}{2}+\frac{i}{2}}$$

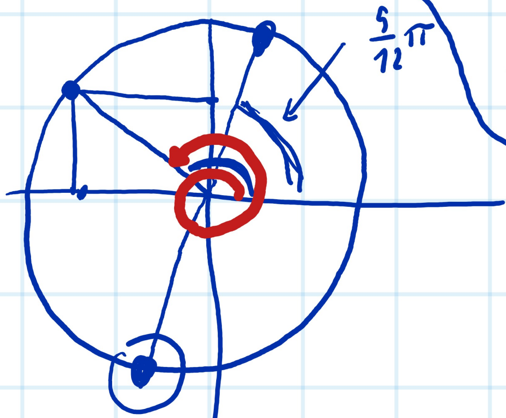
4 $\left((\sqrt{3}-1)+i(\sqrt{3}+1)\right)^{12} =$

$$\left((\sqrt{3}-1)+i(\sqrt{3}+1)\right)^{2} = (\sqrt{3}-1)^2 + \left(i(\sqrt{3}+1)\right)^2 + 2i(\sqrt{3}-1)(\sqrt{3}+1) =$$

$$= 3+1-2\sqrt{3} - (3+1+2\sqrt{3}) + 4i =$$

$$= -4\sqrt{3} + 4i = 8\cdot\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$$

$\arg(\ldots) = \frac{5}{6}\pi$



$\arg(\ldots) = \frac{5}{12}\pi$

$$2\,\frac{17}{12}\pi = \frac{17}{6}\pi = \frac{12}{6}\pi + \frac{5}{6}\pi = 2\pi + \frac{5}{6}\pi$$

$$= \left(\left((\sqrt{3}-1)+i(\sqrt{3}+1)\right)^2\right)^6 =$$

$$= \left(8\cdot\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\right)^6 =$$

$$= 8^6\cdot\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^6 =$$

$$= 2^{18} \cdot \left(\cos\left(\tfrac{5}{6}\pi\right) + \sin\left(\tfrac{5}{6}\pi\right) \cdot i \right)^{6} =$$

$$= 2^{18} \cdot \left(e^{\frac{5}{6}\pi i} \right)^{6} = 2^{18} \cdot \underbrace{e^{5\pi i}}_{\to\, = -1} = -2^{18}$$

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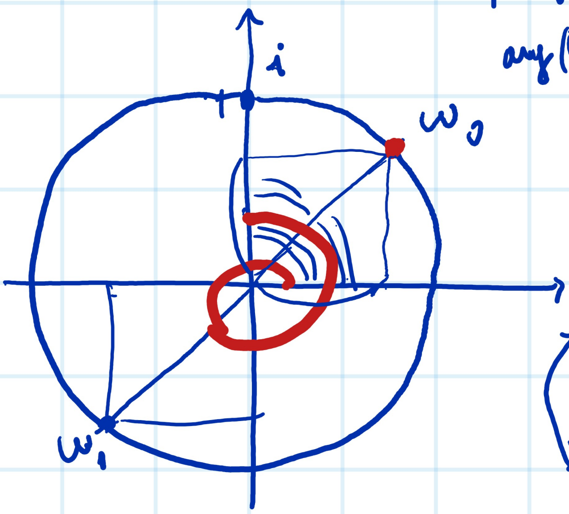
7 $z = i \qquad n = 2, 3, 4$

$|w_0| = 1$

$\arg(w_0) = \frac{\pi}{4}$



$w_0 = e^{\frac{\pi}{4} i}$

$w_1 = e^{\frac{5}{4}\pi i}$

$w_0 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$

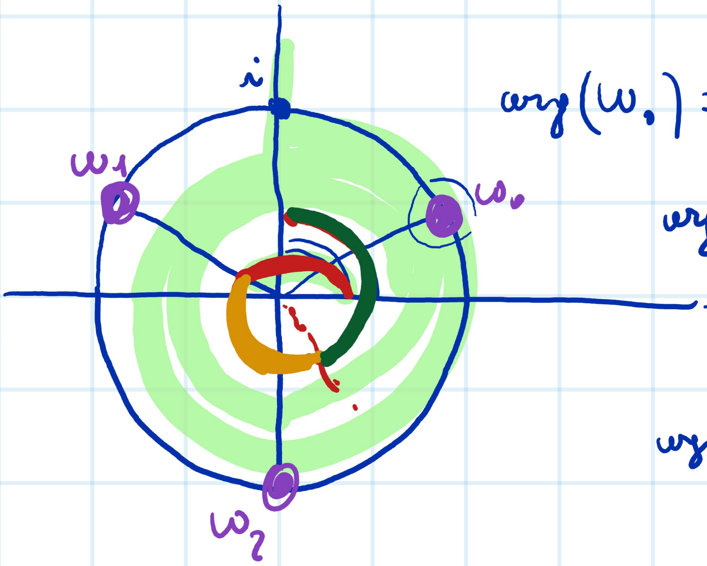
$w_1 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$

$w^3 = i$

$w_0 = e^{\frac{\pi}{6} i} = \frac{\sqrt{3}}{2} + \frac{1}{2} i$

$w_1 = e^{\frac{5}{6}\pi i} = -\frac{\sqrt{3}}{2} + \frac{1}{2} i$

$w_2 = e^{\frac{3}{2}\pi i} = -i$



$\arg(w_0) = \frac{\pi}{6}$

$\arg(w_1) = \frac{\pi}{6} + \frac{2}{3}\pi = \frac{5}{6}\pi$

$\arg(w_2) = \frac{\pi}{6} + \frac{4}{3}\pi = \frac{3}{2}\pi$

10 $z = \dfrac{(5+12i)^4}{(7-24i)^2}$ $n = 8$ $\sqrt[8]{z}$

$(7-24i) = (a+bi)^2$

$= (a+bi)^2 = 7 - 24i$

$= a^2 - b^2 + 2ab\,i$

$(4-3i)^2 = 7-24i$

$$\begin{cases} a^2 - b^2 = 7 \\ 2ab = -24 \end{cases}$$

$a = 4$
$b = -3$

$(-4+3i)^2 = (-1)^2 \cdot (4-3i)^2 =$

$= 7-24i$

$$Z = \frac{(5+12i)^4}{(4-3i)^4} = \left(\frac{(5+12i)(4+3i)}{(4-3i)(4+3i)}\right)^4 =$$

$$= \left(\frac{\overbrace{20-36}^{-16} + 63i}{5^2}\right)^4 =$$

$$= \left(\frac{-16+63i}{5^2}\right)^4 = \left(\frac{63+16i}{5^2}\right)^4 = \ldots$$

$(a+bi)^2 = \boxed{63+16i}$ $\quad (8+i)^2$

$\shortparallel$

a^2-b^2+2abi

$$\begin{cases} a^2-b^2=63 \\ 2ab=16 \end{cases}$$

$(-16+63i)^4 =$

$(63+16i)^4$

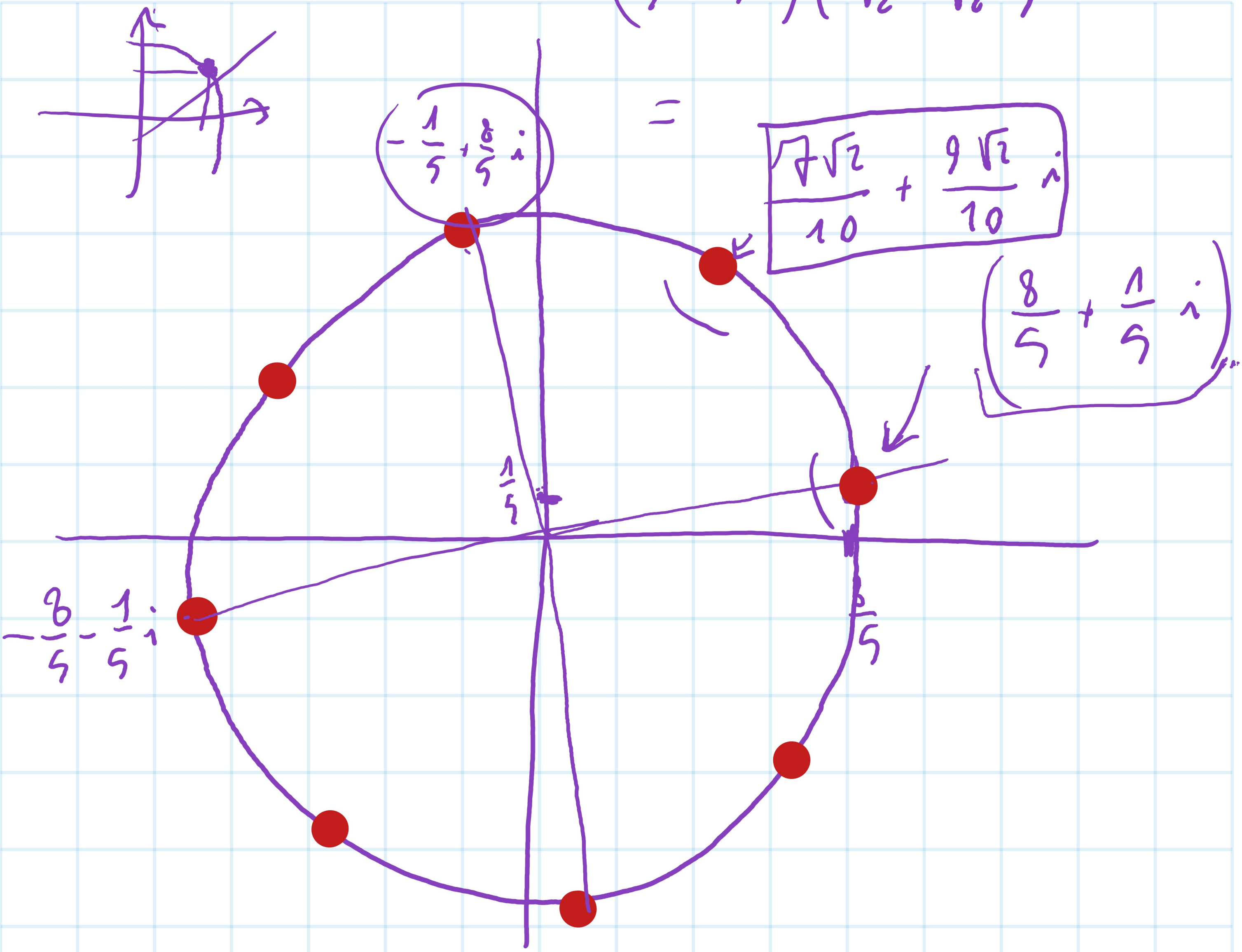
$(-16+63i)^4 \cdot (-i)^4 =$

$= (16i+63)^4$

$$= \left(\frac{(8+i)^2}{5^2}\right)^4 = \boxed{\left(\frac{8+i}{5}\right)^8}$$

$$\left(\frac{8}{5}+\frac{1}{5}i\right)^8$$

$$\left(\frac{8}{5}+\frac{1}{5}i\right)\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)=$$



$$=\boxed{\frac{7\sqrt{2}}{10}+\frac{9\sqrt{2}}{10}i}$$

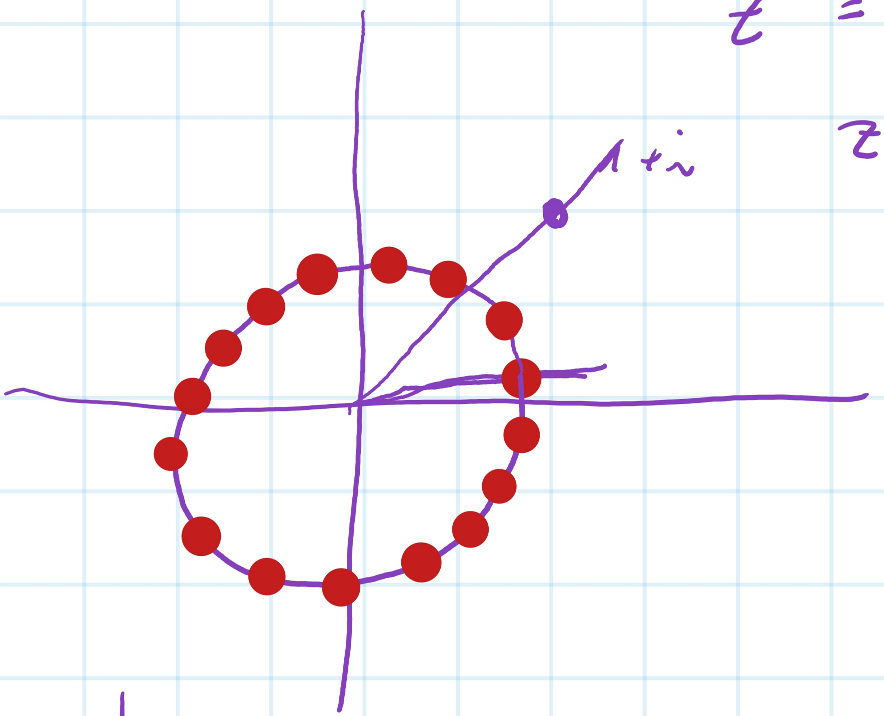
$$\left(\frac{8}{5}+\frac{1}{5}i\right)\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)=\frac{1}{\sqrt{2}}\left(\frac{8}{5}+\frac{1}{5}i\right)(1+i)=$$

$$=\frac{1}{\sqrt{2}}\left(\frac{7}{5}+\frac{9}{5}i\right)=\boxed{\frac{7\sqrt{2}}{10}+\frac{9\sqrt{2}}{10}i}$$

12 LA SOMMA E IL PRODOTTO DI TUTTE LE RADICI 16^e DI $1+i$

$$z^{16} = (1+i)$$

$$z^{16} - (1+i) = 0$$



$$x^2 + bx + c$$

$$(x - x_1)(x - x_2)$$

$$x^2 - (x_1 + x_2)x + x_1 x_2$$

$$z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_1 z + a_0 = 0$$

$$(z - z_1)(z - z_2) \cdots (z - z_n) = 0$$

$$z^n = (1+i)$$

$$z^n - (z_1 + z_2 + \cdots + z_n) z^{n-1}$$

$$z^n - (1+i) = 0$$