Exe 3: Integrale di Riemann (calcolo)

CALCOLARE I SEGUENTI INTEGRALI:

1 $\displaystyle\int_0^{\frac{\pi}{2}} x \sin x \, dx$

2 $\displaystyle\int_0^1 x^2 e^x \, dx$

3 $\displaystyle\int_1^2 \ln x \, dx$

4 $\displaystyle\int_1^{\sqrt{2}} 8x^3 \ln^2 x \, dx$

5 $\displaystyle\int_0^1 4x^3 \arctan x \, dx$

6 $\displaystyle\int_0^{\pi} e^x \sin x \, dx$

7 $\displaystyle\int_1^4 \frac{1}{x+\sqrt{x}} \, dx$

8 $\displaystyle\int_0^1 \sqrt{1-x^2} \, dx$

9 $\displaystyle\int_{\frac{13}{12}}^{\frac{5}{4}} \sqrt{x^2-1} \, dx$

10 $\displaystyle\int_{\frac{5}{12}}^{\frac{3}{4}} \frac{1}{\sqrt{x^2+1}} \, dx$

11 $\displaystyle\int_0^1 \frac{1}{(1+x^2)^2} \, dx$

12 $\displaystyle\int_0^{\pi} x e^x \sin x \, dx$

DI CIASCUNA DELLE SEGUENTI FUNZIONI DIRE SE SONO PERIODICHE, MONOTONE, LIMITATE, CALCOLARNE POI IL LIMITE ℓ PER $x \to 0$ E, SE $\ell = 0$ TROVARNE L'ORDINE DI INFINITESIMO.

13 $\displaystyle F(x) = \int_0^x \cos(\cos t) \, dt$

14 $\displaystyle F(x) = \int_0^x \sin(\sin t) \, dt$

15 $\displaystyle F(x) = \int_0^{x^2} \sin(\sin t) \, dt$

16 $\displaystyle F(x) = \int_x^{x+1} \sin(\sin t) \, dt$

17 $\displaystyle F(x) = \int_{x-x^2}^{x+x^2} \sin(\sin t) \, dt$

18 $\displaystyle \int_0^x \left(\int_0^t \sin(\sin s) \, ds \right) dt$

19 TROVARE INF E SUP SU $[0,+\infty)$ DI $\displaystyle F(x) = \int_{\sqrt{x}}^{\sqrt{1+x}} \sqrt{1+t^2} \, dt.$

SOLUZIONI

① $$\int_0^{\frac{\pi}{2}} x \sin x \, dx = \int_0^{\frac{\pi}{2}} x \cdot (-\cos x)' \, dx = \underbrace{\Big[-x \cos x\Big]_0^{\frac{\pi}{2}}}_{0} + \int_0^{\frac{\pi}{2}} 1 \cdot \cos x \, dx = \Big[\sin x\Big]_0^{\frac{\pi}{2}} = 1$$

② $$\int_0^1 x^2 e^x \, dx = \int_0^1 x^2 \cdot (e^x)' \, dx = \dots$$

③ $$\int_1^2 1 \cdot \ln x \, dx = \int_1^2 x' \cdot \ln x \, dx = \Big[x \ln x\Big]_1^2 - \int_1^2 \cancel{x} \cdot \frac{1}{\cancel{x}} \, dx = 2 \ln 2 - 1$$

④ $$\int_1^{\sqrt{2}} 8x^3 \ln^2 x \, dx = \int_1^{\sqrt{2}} (2x^4)' \cdot \ln^2 x \, dx = \Big[2x^4 \cdot \ln^2 x\Big]_1^{\sqrt{2}} - \int_1^{\sqrt{2}} 2x^{\cancel{4}\,3} \cdot 2 \ln x \cdot \frac{1}{\cancel{x}} \, dx = 8 \cdot \frac{1}{4} \ln^2 2 - \int_1^{\sqrt{2}} (x^4)' \ln x \, dx =$$

$$= 2 \ln^2 2 - \Big[x^4 \ln x\Big]_1^{\sqrt{2}} + \int_1^{\sqrt{2}} x^{\cancel{4}\,3} \cdot \frac{1}{\cancel{x}} \, dx = 2 \ln^2 2 - 2 \ln 2 + \underbrace{\left[\frac{x^4}{4}\right]_1^{\sqrt{2}}}_{\frac{3}{4}}$$

⑤ $$\int_0^1 4x^3 \arctan x \, dx = \int_0^1 (x^4)' \arctan x \, dx = \cancel{\Big[x^4 \arctan x\Big]_0^1 - \int_0^1 x^4 \cdot \frac{1}{1+x^2} \, dx}$$

$$= \int_0^1 (x^4 - 1)' \arctan x \, dx = \underbrace{\cancel{\Big[(x^4-1) \arctan x\Big]_0^1}}_{0} - \int_0^1 \overbrace{(x^4 - 1)}^{\cancel{(x^2+1)}(x^2-1)} \cdot \frac{1}{\cancel{1+x^2}} \, dx =$$

$$= -\int_0^1 (x^2 - 1) \, dx = -\left[\frac{x^3}{3} - x\right]_0^1 = \frac{2}{3}$$

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$$\overbrace{\boxed{\int_0^{\pi} e^x \sin x \, dx}}^{I} = \int_0^{\pi} (e^x)' \cdot \sin x \, dx = \overbrace{\cancel{\left[e^x \sin x\right]_0^{\pi}}}^{0} - \int_0^{\pi} e^x \cos x \, dx = -\int_0^{\pi} (e^x)' \cos x \, dx =$$

$$= -\left[e^x \cdot \cos x\right]_0^{\pi} - \boxed{\int_0^{\pi} e^x \sin x \, dx}$$

$$I = e^{\pi} + 1 - I \qquad 2I = e^{\pi} + 1 \qquad \boxed{I = \frac{e^{\pi}+1}{2}}$$

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$$\overbrace{\boxed{\int_0^{\pi} x e^x \sin x \, dx}}^{J} = \int_0^{\pi} (e^x)' x \sin x \, dx = \cancel{\left[e^x \cdot x \sin x\right]_0^{\pi}} - \int_0^{\pi} e^x (\sin x + x \cos x) \, dx =$$

$$= \underline{-\int_0^{\pi} e^x \sin x \, dx} - \int_0^{\pi} e^x \, x \cos x \, dx =$$

$$= -\frac{e^{\pi}+1}{2} - \int_0^{\pi} (e^x)' x \cos x \, dx = -\frac{e^{\pi}+1}{2} - \left[e^x \cdot x \cos x\right]_0^{\pi} + \int_0^{\pi} e^x (\cos x - x \sin x) \, dx$$

$$= -\frac{e^{\pi}+1}{2} + \pi e^{\pi} + \underbrace{\int_0^{\pi} e^x \cos x \, dx}_{-\frac{e^{\pi}+1}{2}} - \boxed{\int_0^{\pi} x e^x \sin x \, dx}$$

$$J = -(e^{\pi}+1) + \pi e^{\pi} - J \quad \Rightarrow \quad J = \frac{\pi e^{\pi} - e^{\pi} - 1}{2}$$

$$\int_0^1 \frac{1}{(1+x^2)^2}\,dx = \int_0^1 \frac{1+x^2-x^2}{(1+x^2)^2}\,dx = \int_0^1 \frac{\cancel{1+x^2}}{(1+x^2)^{\cancel{2}}} - \frac{x^2}{(1+x^2)^2}\,dx = \underbrace{\int_0^1 \frac{1}{1+x^2}\,dx}_{1} - \underbrace{\int_0^1 \frac{x^2}{(1+x^2)^2}\,dx}_{2} =$$

$$1 = \left[\arctan x\right]_0^1 = \frac{\pi}{4}$$

$$2 = \int_0^1 \frac{1}{2}\,x \cdot \boxed{\frac{2x}{(1+x^2)^2}}\,dx = \int_0^1 \frac{x}{2}\cdot\left(-\frac{1}{1+x^2}\right)'\,dx = \left[\frac{-x}{2(1+x^2)}\right]_0^1 + \int_0^1 \frac{1}{2}\cdot\frac{1}{1+x^2}\,dx =$$

$$= -\frac{1}{4} + \frac{1}{2}\left[\arctan x\right]_0^1 = -\frac{1}{4} + \frac{\pi}{8}$$

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$$\boxed{\int_1^4 \frac{1}{x+\sqrt{x}}\,dx}$$

$$\int_a^b f(x)\,dx = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(t))\,\varphi'(t)\,dt$$

$$t^2-1 \to x$$

$$\int_0^1 \frac{2x}{(1+x^2)^2}\,dx = \int_0^1 \frac{2t}{(1+t^2)^2}\,dt =$$

$$\int_1^2 \frac{1}{\sqrt{1+x}}\,dx =$$

$$= \int_0^1 \frac{1}{(1+t^2)^2}\cdot(\overbrace{1+t^2}^{\varphi(t)})'\,dt =$$

$$= \int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{1+(t^2-1)}}\cdot 2t\,dt =$$

$$= \int_1^2 \frac{1}{x^2}\,dx$$

$$= \int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\cancel{t}}\cdot 2\cancel{t}\,dt = 2(\sqrt{3}-\sqrt{2})$$

$\varphi(t)=t^2 \quad t\geq 0$

$\boxed{\varphi(t)=t^2 \quad t<0}$

$$\int_1^4 \frac{1}{x+\sqrt{x}}\,dx =$$

$$= \int_1^2 \frac{1}{t^2+\sqrt{t^2}}\,2t\,dt = \int_1^2 \frac{1}{t^2+t}\cdot 2t\,dt = \int_1^2 \frac{2}{t+1}\,dt = \Big[2\ln(t+1)\Big]_1^2 = \boxed{2\ln\frac{3}{2}}$$

$$\int_{-1}^{-2} \frac{1}{t^2-t}\,2t = \int_{-1}^{-2} 2\cdot\frac{1}{t-1}\,dt = \Big[2\ln(1-t)\Big]_{-1}^{-2} = 2\ln 3 - 2\ln 2 = 2\ln\left(\frac{3}{2}\right)$$

$$(\ln(t-1))' = \frac{1}{t-1}$$

$$(\ln(1-t))' = \frac{1}{1-t}\cdot(-1) = \frac{1}{t-1}$$

$$\int_1^4 \frac{1}{x+\sqrt{x}}\,dx = \int_1^4 \frac{1}{(\sqrt{x}+1)\cdot\sqrt{x}}\,dx = \int_1^4 \frac{2}{\sqrt{x}+1}\cdot\frac{1}{2\sqrt{x}}\,dx =$$

$$= \int_1^4 \frac{2}{\sqrt{x}+1}\cdot(\sqrt{x}+1)'\,dx = \int_2^3 \frac{2}{y}\,dy = \Big[2\ln y\Big]_2^3 = 2\ln\frac{3}{2}$$

$y \leftarrow (\sqrt{x}+1)$

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$t\geq 0$, $x = ch\,t$

$$\int_{\frac{13}{12}}^{\frac{5}{4}} \sqrt{x^2-1}\,dx =$$

$$= \boxed{\int_{\ln\frac{3}{2}}^{\ln 2} \sqrt{ch^2 t - 1}\; sh\,t\,dt} = (*)$$

$ch\,t = \frac{5}{4} \qquad t = ?$

$$\frac{e^t+e^{-t}}{2} = \frac{5}{4}$$

$$y = sh\,x = \frac{e^x - e^{-x}}{2}$$

$$x = ch\,x = \frac{e^x + e^{-x}}{2}$$

$$(*) \quad \boxed{x^2 - y^2} = ch^2 x - sh^2 x = \frac{\cancel{e^{2x}}+\cancel{e^{-2x}}+2}{4} - \frac{e^{2x}+e^{-2x}-2}{4} = 1$$

$$(sh\,x)' = \left(\frac{e^x-e^{-x}}{2}\right)' = \frac{e^x+e^{-x}}{2} = ch\,x$$

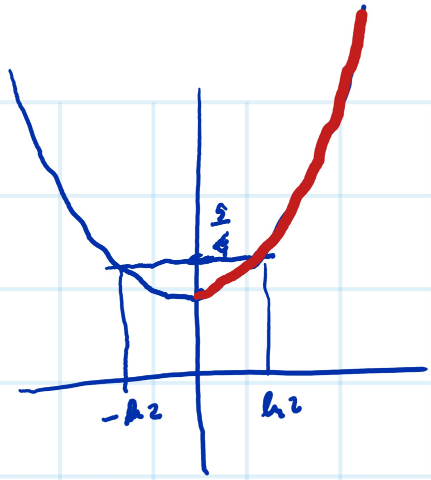
$$(ch\,x)' = \left(\frac{e^x+e^{-x}}{2}\right)' = \frac{e^x-e^{-x}}{2} = sh\,x$$

$e^t + \frac{1}{e^t} = \frac{5}{2}$

$(e^t)^2 + 1 = \frac{5}{2} e^t$

$e^t = 2 \Rightarrow t = \ln 2$

$e^t = \frac{1}{2}$ $\quad t = -\ln 2$



$2(e^t)^2 - 5e^t + 2 = 0 \qquad y = e^t$

$2y^2 - 5y + 2 = 0$

$y = \frac{5 \pm \sqrt{25-16}}{4} = \frac{5 \pm 3}{4} =$

$\operatorname{ch} t = \frac{13}{12}$

$e^t + e^{-t} = \frac{13}{6}$

$6(e^t)^2 - 13e^t + 6 = 0$

$e^t = \frac{13 \pm \sqrt{169-144}}{12} = \frac{13 \pm 5}{12} = \begin{cases} \frac{2}{3} \\ \frac{3}{2} \end{cases}$

$\ln \frac{2}{3} = -\ln\left(\frac{3}{2}\right)$

$\ln \frac{3}{2}$

$$(*) = \int_{\ln \frac{3}{2}}^{\ln 2} \operatorname{sh}^2 t \, dt = \frac{1}{4} \int_{\ln \frac{3}{2}}^{\ln 2} (e^{2t} + e^{-2t} - 2)\, dt = \frac{1}{4} \left[\frac{1}{2} e^{2t} - \frac{1}{2} e^{-2t} - 2t \right]_{\ln \frac{3}{2}}^{\ln 2} =$$

$$\operatorname{sh}^2 t = \left(\frac{e^t - e^{-t}}{2} \right)^2 = \frac{e^{2t} + e^{-2t} - 2}{4}$$

$$= \frac{1}{4}\left(\frac{1}{2} \cdot 4 - \frac{1}{2} \cdot \frac{1}{4} - 2\ln 2 - \frac{1}{2} \frac{9}{4} + \frac{1}{2} \frac{4}{9} + 2\ln \frac{3}{2} \right) = \dots$$

10 $\displaystyle \int_{\frac{5}{12}}^{\frac{3}{4}} \sqrt{x^2+1}\, dx \overset{x = \operatorname{sh} t}{=} \int \sqrt{\operatorname{sh}^2 t + 1} \cdot \operatorname{ch} t \, dt = \int \operatorname{ch}^2 t \, dt \dots$