Esercitazioni di Analisi Matematica 2 - 27 Marzo 2024 - docente: Callegari

Exe 4: Integrali impropri (1)

CALCOLARE, SE CONVERGONO, I SEGUENTI INTEGRALI IMPROPRI

1 $$\int_0^{+\infty} \frac{3x}{x^4+5x^2+4}\,dx$$

2 $$\int_1^{+\infty} \frac{3}{x^2\cdot\sqrt{x}+x}\,dx$$

3 $$\int_{-\infty}^{+\infty} \frac{x}{1+x^4}\,dx$$

4 $$\int_{-\infty}^{+\infty} \frac{x}{1+x^2}\,dx$$

5 $$\int_0^{+\infty} \frac{2x\ln(1+x^2)}{(2+x^2)^2}\,dx$$

6 $$\int_{-\infty}^{+\infty} \ln\left(\frac{x^2+8x+20}{x^2+8x+17}\right)dx$$

7 $$\int_{-\infty}^{+\infty} |x|\ln\left(\frac{x^4+5x^2+6}{x^4+5x^2+4}\right)dx$$

8 $$\int_{-2}^{2} \ln\left(\frac{1}{\sqrt[8]{4-x^2}}\right)dx$$

STUDIARE IL CARATTERE DEI SEGUENTI INTEGRALI IMPROPRI

9 $$\int_0^{+\infty} \frac{\sin x^4+\cos x^4}{\ln(1+e^x)}\,dx$$

10 $$\int_{\frac{1}{400}}^{+\infty} \sin\left(\frac{1}{x^2+\sin x}\right)dx$$

11 $$\int_0^{+\infty} \frac{2x+2-\sqrt{4+2x}}{(x(x+2))^{\alpha}}\,dx \quad (\text{con } \alpha>0)$$

12 $$\int_0^{+\infty} \frac{1}{x}\sin\frac{1}{x}\,dx$$

13 $$\int_0^{+\infty} \frac{\arctan(\sin x)}{\ln(e+x+\sin x)}\,dx$$

14 $$\int_0^{+\infty} \frac{1}{x}\ln\left(1+\frac{1}{2}\sin x\right)dx$$

15 $$\int_0^{+\infty} \frac{\sin x^2}{x}\,dx$$

16 $$\int_0^{+\infty} \frac{\sin x}{x+\cos x^2}\,dx$$

17 $$\int_0^{+\infty} \left(1+\frac{1}{x^2}\right)^x - e^{\frac{1}{x}}\,dx$$

SOLUZIONI

1
$$\int_0^{+\infty} \frac{3x}{x^4+5x^2+4}\,dx = \lim_{b\to+\infty} \frac{3}{2}\int_0^b \frac{2x}{x^4+5x^2+4}\,dx =$$

$$= \lim_{b\to+\infty} \frac{3}{2}\int_0^b \frac{1}{x^4+5x^2+4}\cdot(x^2)'\,dx \overset{y=x^2}{=} \lim_{b\to+\infty}\frac{1}{2}\int_0^{b^2} \frac{3}{y^2+5y+4}\,dy =$$

$$\frac{y+4-1-y}{(y+1)(y+4)} = \frac{1}{y+1}-\frac{1}{y+4}$$

$$= \lim_{c\to+\infty} \frac{1}{2}\int_0^c \left(\frac{1}{y+1}-\frac{1}{y+4}\right)dy = \lim_{c\to+\infty}\frac{1}{2}\left[\ln\frac{y+1}{y+4}\right]_0^c =$$

$$= \lim_{c\to+\infty}\frac{1}{2}\left(\ln\frac{c+1}{c+4}-\ln\frac{1}{4}\right) = \ln 2$$

NO

$$\lim_{c\to+\infty}\frac{1}{2}\int_0^c\frac{1}{y+1}\,dy - \lim_{c\to+\infty}\frac{1}{2}\int_0^c\frac{1}{y+4}$$

2
$$\int_1^{+\infty}\frac{3}{x^2\cdot\sqrt{x}+x}\,dx = \lim_{b\to+\infty}\int_1^b \frac{3}{x^2\sqrt{x}+x}\,dx \overset{x=t^2}{=} \lim_{b\to+\infty}\int_1^{\sqrt{b}} \frac{3}{t^4\cdot t+t^2}(2t)\,dt =$$

LUNGA

$$\frac{6}{t^4+t} = \frac{6}{t(t+1)(t^2-t+1)}$$

$$= \lim_{b\to+\infty}\int_1^{\sqrt{b}}\frac{2}{t^3(t^3+1)}\underbrace{(3t^2)}_{(t^3)'}\,dt =$$

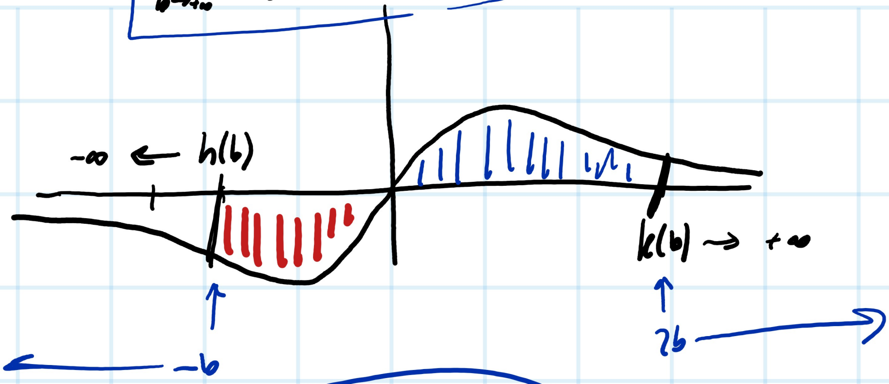
$y=t^3$

$$= \lim_{\substack{b\to+\infty\\ c\to+\infty}} 2\int_1^{b\sqrt{b}\ (c)} \frac{1}{y(y+1)}\,dy = \lim_{c\to+\infty} 2\int_1^c \frac{1+y-y}{y(y+1)}\,dy =$$

$$= \lim_{c\to+\infty} 2\int_1^c\left(\frac{1}{y}-\frac{1}{y+1}\right)dy = \lim_{c\to+\infty} 2\left[\ln\frac{y}{y+1}\right]_1^c = \lim_{c\to+\infty} 2\left(\ln\frac{c}{c+1}-\ln\frac{1}{2}\right) = 2\ln 2$$

4 $\int_{-\infty}^{+\infty} \frac{x}{1+x^2}\,dx$

$$\boxed{\lim_{b\to+\infty}\int_{-b}^{b}\frac{x}{1+x^2}\,dx = \lim_{b\to+\infty} 0 = 0}$$



NO

$$\lim_{b\to+\infty}\int_{-b}^{2b}\frac{1}{2}\cdot\frac{2x}{1+x^2}\,dx = \lim_{b\to+\infty}\left[\frac{1}{2}\ln(1+x^2)\right]_{-b}^{2b} =$$

$$= \lim_{b\to+\infty}\left|\frac{1}{2}\ln\left(\frac{1+4b^2}{1+b^2}\right)\right| = \ln 2$$

$$\int_{-\infty}^{+\infty}\frac{x}{1+x^2}\,dx = \int_{0}^{+\infty}\frac{x}{1+x^2}\,dx + \int_{-\infty}^{0}\frac{x}{1+x^2}\,dx =$$

$$= \lim_{b\to+\infty}\int_{0}^{b\,(2b)}\frac{x}{1+x^2}\,dx + \lim_{b\to+\infty}\int_{-b}^{0}\frac{x}{1+x^2}\,dx =$$

$$= \underbrace{\lim_{b\to+\infty}\ln(1+b^2)}_{+\infty} - \underbrace{\lim_{b\to+\infty}\ln(1+b^2)}_{-\infty}$$

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3 $\int_{-\infty}^{+\infty}\frac{x}{1+x^4}\,dx = \lim_{b\to+\infty}\frac{1}{2}\int_{h(b)}^{k(b)}\frac{1}{1+x^4}\cdot(x^2)'\,dx =$

$k(b)\to+\infty$

$h(b)\to-\infty$

$$\lim_{b\to+\infty}\frac{1}{2}\int_{h(b)}^{0}\frac{1}{1+x^4}(x^2)'\,dx + \lim_{b\to+\infty}\frac{1}{2}\int_{0}^{k(b)}\frac{1}{1+x^4}\cdot(x^2)'\,dx =$$

$$= \lim_{b\to+\infty} \frac{1}{2}\int_{(\ln(b))^2}^{0} \frac{1}{1+y^2}\,dy + \lim_{b\to+\infty} \frac{1}{2}\int_{0}^{(\ln(b))^2} \frac{1}{1+y^2}\,dy =$$

$$= \lim_{b\to+\infty} -\frac{1}{2}\int_{0}^{(\ln(b))^2} \frac{1}{1+y^2}\,dy \;\ldots$$

$$= \lim_{b\to+\infty} -\frac{1}{2}\Big[\arctan\Big]_0^{(\ln(b))^2} + \lim_{b\to+\infty} \frac{1}{2}\Big[\arctan\Big]_0^{(\ln(b))^2} =$$

$$= \lim_{b\to+\infty} -\frac{1}{2}\arctan\left((\ln(b))^2\right) + \lim_{b\to+\infty} \frac{1}{2}\arctan\left((\ln(b))^2\right) =$$

$$= -\frac{\pi}{4} + \frac{\pi}{4} = 0$$

5

$$y = 1+x^2$$

$$\int_0^{+\infty} \frac{2x\ln(1+x^2)}{(2+x^2)^2}\,dx = \lim_{b\to+\infty}\int_0^b \frac{\ln(1+x^2)}{\left(1+(1+x^2)\right)^2}(1+x^2)'\,dx =$$

$$= \lim_{b\to+\infty}\int_1^{1+b^2} \frac{\ln(y)}{(1+y)^2}\,dy = \lim_{b\to+\infty}\int_1^{1+b^2 = c} \left(-\frac{1}{1+y}\right)'\cdot\ln y\,dy =$$

$$= \lim_{c\to+\infty}\left(\left[-\frac{1}{1+y}\cdot\ln y\right]_1^c + \int_1^c \frac{1+y-y}{(1+y)\cdot y}\,dy\right) =$$

$$= \lim_{c\to+\infty}\left(-\underbrace{\frac{\ln c}{1+c}}_{\to 0} + \underbrace{\Big[\ln y - \ln(1+y)\Big]_1^c}_{\ln\frac{y}{1+y}}\right) = \boxed{\ln 2}$$

$$\to \underbrace{\ln\left(\frac{c}{c+1}\right)}_{\to 0} - \ln\frac{1}{2}$$

$\boxed{7}$ $\displaystyle\int_{-\infty}^{+\infty} |x| \ln\left(\frac{x^4+5x^2+6}{x^4+5x^2+4}\right) dx = 2\int_0^{+\infty} \cdots = \lim_{b\to+\infty} \int_0^b \underbrace{2x}_{(x^2)'} \ln\left(\frac{x^4+5x^2+6}{x^4+5x^2+4}\right) dx =$

$x^2 = y$

$\displaystyle = \lim_{\substack{b\to+\infty \\ c\to+\infty}} \int_0^{c} \ln\left(\frac{y^2+5y+6}{y^2+5y+4}\right) dy = \lim_{c\to+\infty} \int_0^c \ln(y+2)+\ln(y+3)-\ln(y+1)-\ln(y+4)\, dy$

$\displaystyle \frac{y^2+5y+6}{y^2+5y+4} = \frac{(y+2)(y+3)}{(y+1)(y+4)}$

$\displaystyle = \lim_{c\to+\infty} \left(\int_0^c (y+2)' \ln(y+2)\, dy + \int_0^c \ln(y+3)\, dy - \int_0^c \ln(y+1)\, dy - \int_0^c \ln(y+4)\, dy \right) =$

$\displaystyle = \lim_{c\to+\infty} \left(\Big[(y+2)\ln(y+2)\Big]_0^c - \int_0^c \cancel{(y+2)} \cdot \frac{1}{\cancel{y+2}}\, dy \quad \ldots \ldots \ldots \right) =$

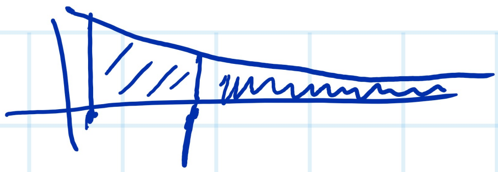
$\displaystyle = \lim_{c\to+\infty} \Big((c+2)\ln(c+2) - 2\ln 2 - \cancel{c} + (c+3)\ln(c+3) - 3\ln 3 - \cancel{c} - (c+1)\ln(c+1) + 1\ln 1 + \cancel{c} - (c+4)\ln(c+4) + 4\ln 4 + \cancel{c} \Big)$

$\displaystyle = \lim_{c\to+\infty} \left(c \ln \frac{\overbrace{(c+2)(c+3)}^{c^2+5c+6}}{\underbrace{(c+1)(c+4)}_{c^2+5c+4}} + \underbrace{\ln \frac{\overbrace{(c+2)^2(c+3)^3}^{c^5+\cdots}}{\underbrace{(c+1)(c+4)^4}_{c^5+\cdots}}}_{\to 0} \underbrace{-2\ln 2 - 3\ln 3 + 1\ln 1 + 4\ln 4}_{\boxed{6\ln 2 - 3\ln 3}} \right) =$

$\displaystyle \frac{c^2+5c+4+2}{c^2+5c+4}$

$\displaystyle \lim_{c\to+\infty} \underbrace{\left(c \cdot \ln\left(1 + \frac{2}{c^2+5c+4}\right)\right)}$

$\displaystyle c \cdot \frac{2}{c^2+5c+4} \to 0$



9 $\displaystyle\int_{\boxed{0}\ \boxed{1}}^{+\infty} \frac{\sin^4 x + \cos^4 x}{\ln(1+e^x)}\, dx$

$m = \min(g) > 0$

$$\frac{\sin^4 x + \cos^4 x}{\ln(1+e^x)} \geq \frac{m}{\ln(1+e^x)} \approx \boxed{\frac{m}{x}}$$

Pour $x \to +\infty$

$$\int_1^{+\infty} \frac{m}{x}\, dx = m \int_1^{+\infty} \frac{1}{x}\, dx \quad \boxed{\text{DIV}}$$

$F(x) = \int_0^x \sin(t^2)\, dt$

15 $\displaystyle\to \int_0^{+\infty} \frac{\sin x^2}{\ln(1+e^{x+\sin x})}\, dx = \int_0^{+\infty} \boxed{\sin(x^2)} \cdot \boxed{\frac{1}{\ln(1+e^{x+\sin x})}}\, dx$

$\to 0$ DECR.

$\displaystyle\int_0^{+\infty} \sin(x^2)\, dx$ Converge

$\displaystyle\int_0^{+\infty} \sin(x^2)\, dx$

$$\int_{\boxed{0}\ \boxed{1}}^{+\infty} \sin(x^2)\, dx = \lim_{b\to+\infty} \int_1^b \sin(x^2)\, dx \overset{x=\sqrt{t}}{=} \lim_{b\to+\infty} \int_1^{\sqrt{b}} \sin(t) \cdot \frac{1}{2\sqrt{t}}\, dt =$$

$$= \int_1^{+\infty} \frac{\sin t}{2\sqrt{t}}\, dt = \int_1^{+\infty} (\sin t) \cdot \frac{1}{2\sqrt{t}}\, dt \quad \text{conv.}$$