Esercitazioni di Analisi Matematica 2 - 28 Marzo 2024 - docente: Callegari

Exe 5: Integrali impropri (2)

[CONTINUA DA LEZIONE SCORSA

STUDIARE IL CARATTERE DEI SEGUENTI INTEGRALI IMPROPRI

~~9~~ $\displaystyle\int_0^{+\infty} \frac{\sin x^4 + \cos x^4}{\ln(1+e^x)}\,dx$

10 $\displaystyle\int_{\frac{1}{400}}^{+\infty} \sin\left(\frac{1}{x^2+\sin x}\right) dx$

11 $\displaystyle\int_0^{+\infty} \frac{2x+2-\sqrt{4+2x}}{\left(x(x+2)\right)^{\alpha}}\,dx$ (con $\alpha>0$)

12 $\displaystyle\int_0^{+\infty} \frac{1}{x}\sin\frac{1}{x}\,dx$

13 $\displaystyle\int_0^{+\infty} \frac{\arctan(\sin x)}{\ln(e+x+\sin x)}\,dx$

14 $\displaystyle\int_0^{+\infty} \frac{1}{x}\ln\left(1+\frac{1}{2}\sin x\right) dx$

~~15~~ $\displaystyle\int_0^{+\infty} \frac{\sin x^2}{x}\,dx$

16 $\displaystyle\int_0^{+\infty} \frac{\sin x}{x+\cos x^2}\,dx$

17 $\displaystyle\int_0^{+\infty} \left(1+\frac{1}{x^2}\right)^x - e^{\frac{1}{x}}\,dx$

18 $\displaystyle\int_0^{+\infty} \left|\alpha+\frac{\sin x}{4}\right|^x dx$ (con $\alpha>0$)

19 $\displaystyle\int_0^{1} \left(\frac{e^x-\cos x}{\tan x-\sin x}\right)^{\alpha} dx$ (con $\alpha>0$)

20 $\displaystyle\int_0^{+\infty} |\sin(\sin x)|^x\,dx$

21 $\displaystyle\int_0^{+\infty} \left(\frac{A+\sin x}{17+\cos x}\right)^x dx$ (con $A>0$)

STUDIARE LE SEGUENTI FUNZIONI:

22 $\displaystyle F(x)=\int_2^x \frac{e^{\frac{1}{t}}}{t\sqrt{t^2+t-2}}\,dt$

SOLUZIONI

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$$\int_0^{+\infty} \frac{\arctan(\sin x)}{\ln(e+x+\sin x)}\,dx = \int_0^{+\infty} \boxed{\arctan(\sin x)} \cdot \boxed{\frac{1}{\ln(e+\underbrace{x+\sin x})}}\,dx$$

$$\int_{\text{Periodo}} \arctan(\sin x)\,dx = 0$$

$g(x) = x + \sin x$

$g'(x) = 1 + \cos x \geq 0$

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$$\underbrace{\int_0^{+\infty} \frac{1}{x} \cdot \ln\left(1+\underbrace{\tfrac{1}{2}\sin x}\right) dx}_{} = \int_0^{+\infty} \boxed{\ln\left(1+\frac{1}{2}\sin x\right)} \cdot \boxed{\frac{1}{x}}\,dx$$

$\frac{1}{x}$ DECRESCENDO $\to 0$

$$\ln(1+y) > 0 \iff y > 0$$

$f(\sin x)$

Periodiche a media nulla hanno primitiva limitata

$f(x)$ = periodica di periodo T t.c. $\int_0^T f(x)\,dx = 0$

$$F(x) = \int_0^x f(t)\,dt$$

$$F(x) \overset{?}{=} \boxed{F(x+T)} = \int_0^{x+T} f(t)\,dt = \int_0^x f(t)\,dt + \int_x^{x+T} f(t)\,dt = F(x) + 0 = \boxed{F(x)}$$

$\forall x \in \mathbb{R}\quad F(x+T) = F(x)$

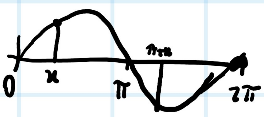
$$\boxed{f(x) = 1 + \sin x}$$

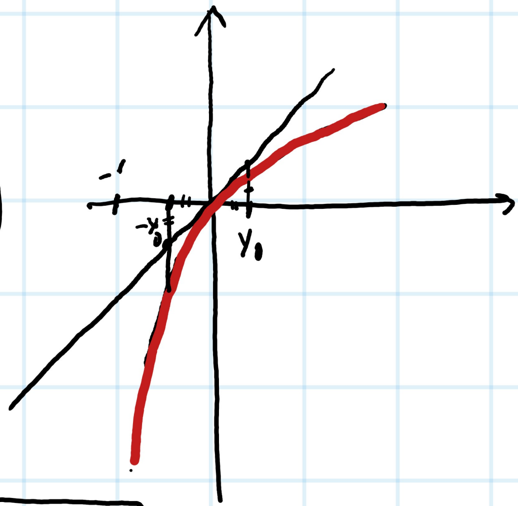
$$\int_0^x f(t)\,dt = \int_0^x (1 + \sin t)\,dt = \Big[t - \cos x\Big]_0^x = x - \cos x + 1 =$$

$$= x + \boxed{1 - \cos x}$$

$$\boxed{\int_0^{2\pi} \ln\left(1 + \frac{1}{2}\sin x\right) dx} =$$

$\ln(1+y)$





$$= \int_0^{\pi} \ln\left(1 + \frac{1}{2}\sin x\right) dx + \int_{\pi}^{2\pi} \ln\left(1 + \frac{1}{2}\sin x\right) dx =$$

$$= \int_0^{\pi} \ln\left(1 + \frac{1}{2}\sin(y+\pi)\right) \cdot 1\,dy =$$

$y = x - \pi$

$x = y + \pi$

$$= \int_0^{\pi} \ln\left(1 + \frac{1}{2}\sin(x+\pi)\right) dx$$

$$= \int_0^{\pi} \ln\left(1+\frac{1}{2}\sin x\right) + \ln\left(1+\frac{1}{2}\overbrace{\sin(x+\pi)}^{-\sin x}\right) dx =$$

$$\ln\left(1-\frac{1}{2}\sin x\right) + \ln\left(1+\frac{1}{2}\sin x\right) =$$

$$= \ln\left(1-\frac{1}{4}\sin^2 x\right) < 0$$

$$= \int_0^{\pi} \ln\left(1-\frac{1}{4}\sin^2 x\right) dx = C < 0$$

$$\boxed{\int_0^{+\infty} f(x)\,dx} \;??$$

$$\int_0^{+\infty} g(x)\,dx \quad \text{CONVERGE}$$

$$\boxed{\int_0^{+\infty} f(x)-g(x)\,dx}$$

$$\underbrace{f(x)}_{③} = \underbrace{\left(f(x)-g(x)\right)}_{①} + \underbrace{g(x)}_{②}$$

$\int ①$ converge $\Rightarrow \int ③$ converge

OSS. (TRUCCO)

DATE $f(x)$, $g(x)$ SU $[a,+\infty)$ t.c. $\int_a^{+\infty} g(x)\,dx$ converge, ALLORA

$\int_a^{+\infty} f(x)\,dx$ $\quad \int_a^{+\infty} f(x)-g(x)\,dx$ HANNO STESSO CARATTERE

DIM

(1) $\int_a^{+\infty} f(x)-g(x)\,dx$ CONVERGE $\Rightarrow$ $\int_a^{+\infty} f(x)\,dx$ CONVERGE

INDET

DIVERGE $+\infty$ $\qquad$ DIVERGE $+\infty$

$\lim_{b\to+\infty} \boxed{\int_a^b f(x)-g(x)\,dx}$ $\exists$ finito $\quad$ $\nexists$ $+\infty$

$\lim_{b\to+\infty} \boxed{\int_a^b g(x)\,dx}$ $\boxed{\exists \text{ finito}}$

$\Rightarrow \lim_{b\to+\infty}\left(\int_a^b f(x)-g(x)\,dx + \int_a^b g(x)\,dx\right)$ $\exists$ finito $\quad$ $\nexists$ $(+\infty)$

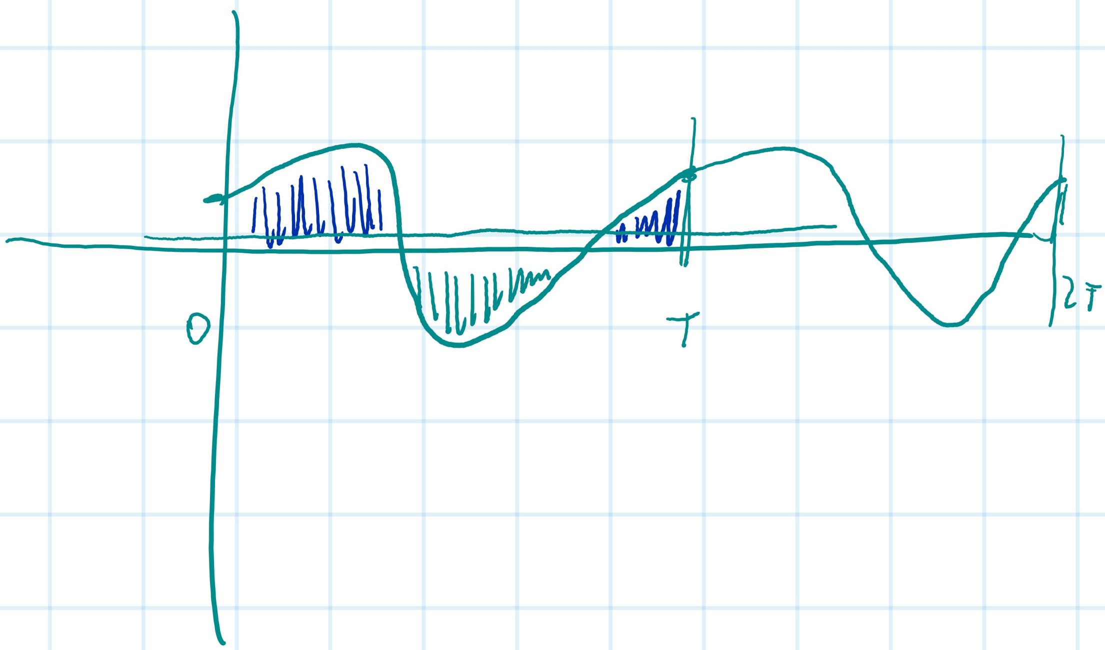
$\int_a^b f(x)\,dx$ $\quad +\infty$

$\Downarrow$

$\int_a^{+\infty} f(x)\,dx$ converge

INDET.

(DIVERGE A $+\infty$)



$$\boxed{f(x) = \ln\left(1 + \frac{1}{2}\sin x\right)} \qquad \int_0^{+\infty} \frac{1}{x} \cdot f(x)\, dx \qquad \int_0^{2\pi} f(x)\, dx = C < 0$$

$$g(x) = f(x) - \frac{C}{2\pi}$$

$$\int_0^{2\pi} g(x)\,dx = \int_0^{2\pi} f(x) - \frac{C}{2\pi}\, dx = \int_0^{2\pi} f(x)\,dx - \int_0^{2\pi} \frac{C}{2\pi}\, dx = \cancel{C} - \cancel{C} = 0$$

$$\int_0^{+\infty} \frac{1}{x}\, g(x)\, dx$$ CONVERGE PER CR. INT. OSC.

$$\int_0^{+\infty} \frac{1}{x} f(x)\, dx$$ HA STESSO CARATTERE DI $$\underbrace{\int_0^{+\infty} \frac{1}{x} f(x) - \frac{1}{x} \underbrace{g(x)}_{f(x) - \frac{C}{2\pi}}\, dx}$$ =

$$= \int_0^{+\infty} \frac{1}{x}\left(\cancel{f(x)} - \cancel{f(x)} + \frac{C}{2\pi}\right) dx = \int_0^{+\infty} \frac{\frac{C}{2\pi}}{x}\, dx =$$

$$= \frac{C}{2\pi} \int_0^{+\infty} \frac{1}{x}\, dx$$

DIVERGE

$$\int_a^{+\infty} \underbrace{|f(x)|}_{C} \cdot \underbrace{b(x)}_{\downarrow 0 \text{ DECR.}} \, dx$$

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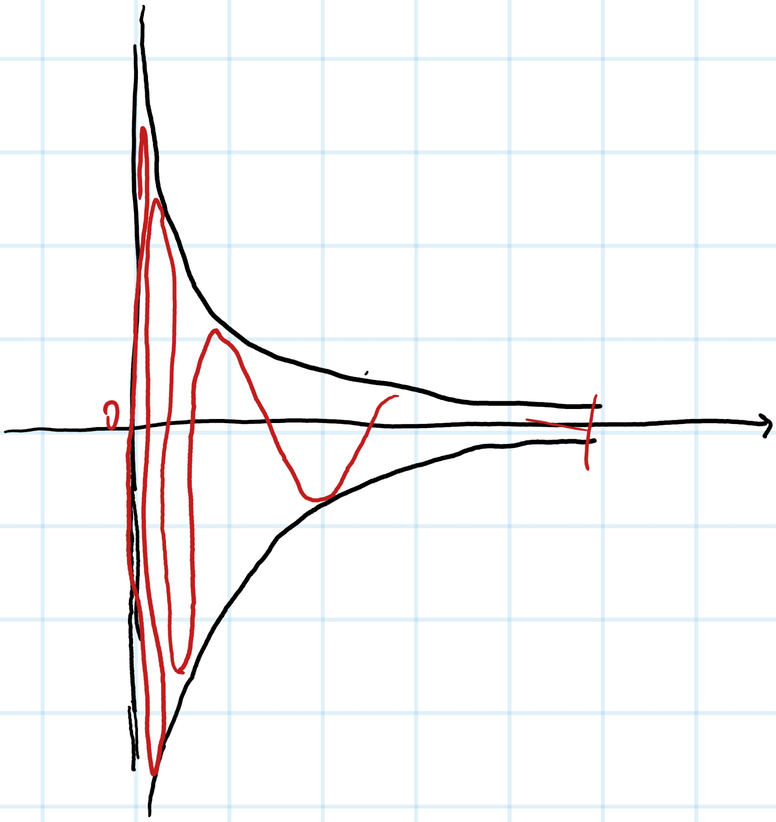
$$\int_0^{+\infty} \frac{1}{x} \sin\frac{1}{x} \, dx$$



$$\int_0^{+\infty} \frac{1}{x} \sin\frac{1}{x} \, dx = \overbrace{\boxed{\int_0^1 \frac{1}{x} \sin\frac{1}{x} \, dx}}^{\text{CONV}} + \overbrace{\int_1^{+\infty} \underbrace{\frac{1}{x} \sin\frac{1}{x}}_{} \, dx}^{\text{CONV}}$$

$$\int_1^{+\infty} \frac{1}{x^2} \, dx$$

$$\frac{1}{x} \cdot \sin\frac{1}{x} \simeq \frac{1}{x} \cdot \frac{1}{x} = \frac{1}{x^2}$$

$$\lim_{b \to 0^+} \int_b^1 \frac{1}{x} \sin\frac{1}{x} \, dx = \lim_{b \to 0^+} \int_{\frac{1}{b}}^{1} \left(\frac{1}{\frac{1}{t}}\right) \cdot \sin\left(\frac{1}{t}\right) \cdot \left(-\frac{1}{t^2}\right) dt =$$

$x = \frac{1}{t}$

$$= \lim_{b\to 0^+} \int_{\frac{1}{b}}^{1} \sin t \cdot \left(-\frac{1}{t}\right) dt = \lim_{b\to 0^+} \int_{1}^{\frac{1}{b}} \frac{\sin t}{t}\, dt =$$

$$= \lim_{c\to +\infty} \int_1^c \frac{\sin t}{t}\, dt = \boxed{\int_1^{+\infty} \frac{\sin t}{t}\, dt}$$

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18 $\displaystyle\int_0^{+\infty} \left|\alpha + \frac{\sin x}{4}\right|^x dx \quad (\text{con } \alpha > 0)$

$\alpha = 2 \quad \alpha = \frac{1}{2} \quad \alpha = 1 \quad \boxed{\alpha = \frac{3}{4}}$ FAC.

$$\int_0^{+\infty} \underbrace{\left|2 + \frac{\sin x}{4}\right|^x}_{\geq \left(\frac{3}{2}\right)^x > 1} dx$$

$$\int_0^{+\infty} \left|\frac{1}{2} + \frac{\sin x}{4}\right|^x dx$$

$\leq \boxed{\left(\frac{3}{4}\right)^x} \underset{\text{def.}}{\leq} \boxed{\frac{1}{x^2}}$

$\int_1$

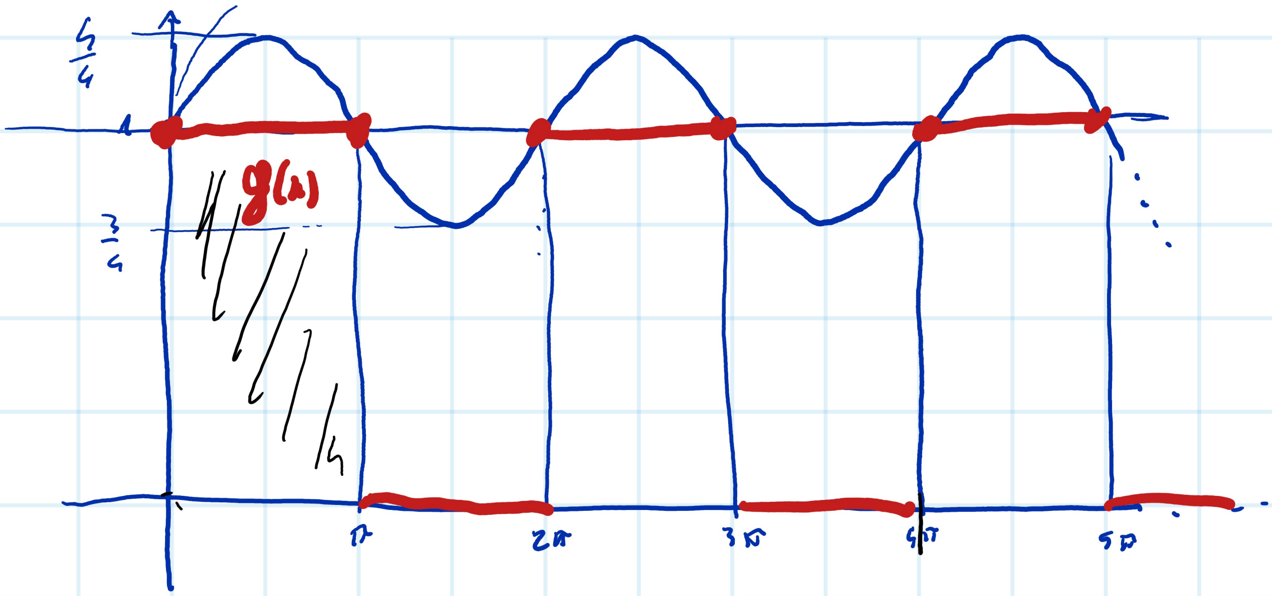
$\frac{1}{4} \leq \bullet \leq \frac{3}{4}$

$$-\frac{1}{4} \leq \frac{\sin x}{4} \leq \frac{1}{4}$$

$$\frac{1}{4} \leq \frac{\sin x}{4} + \frac{1}{2} \leq \frac{3}{4}$$

$\boxed{\alpha = 1}$

$$\int_0^{+\infty} \left(1 + \frac{\sin x}{4}\right)^x dx$$



$$\int_0^{2k\pi} f(x)\,dx = k\pi$$

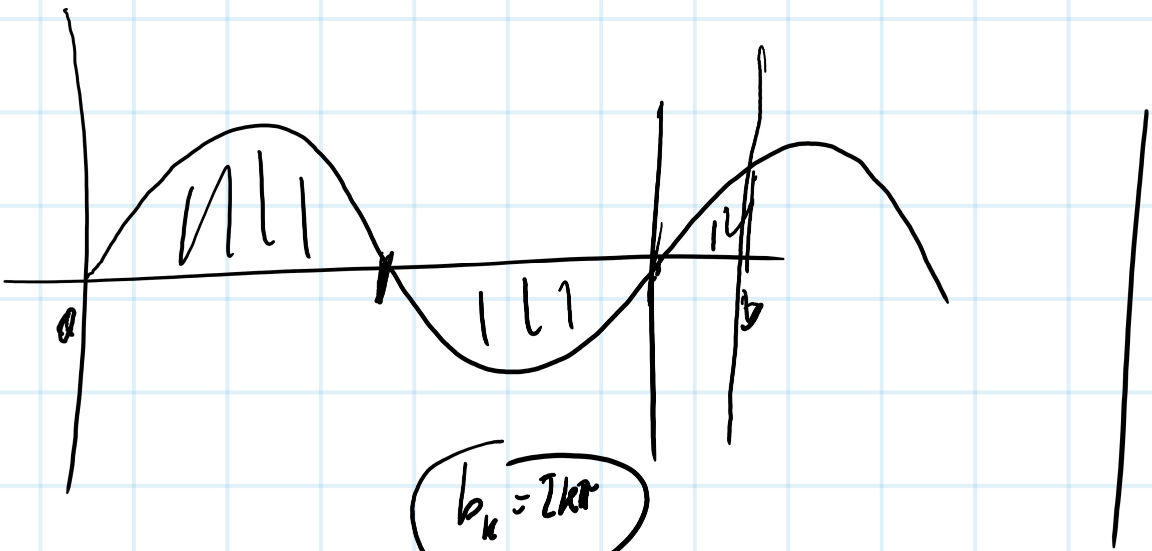
$$\lim_{b\to+\infty}\int_a^b f(x)\,dx =$$

$$\lim_{k\to+\infty}\int_a^{b_k} f(x)\,dx =$$

(con $b_k \to +\infty$)

$$\lim_{b\to+\infty}\int_0^b f(x)\,dx$$

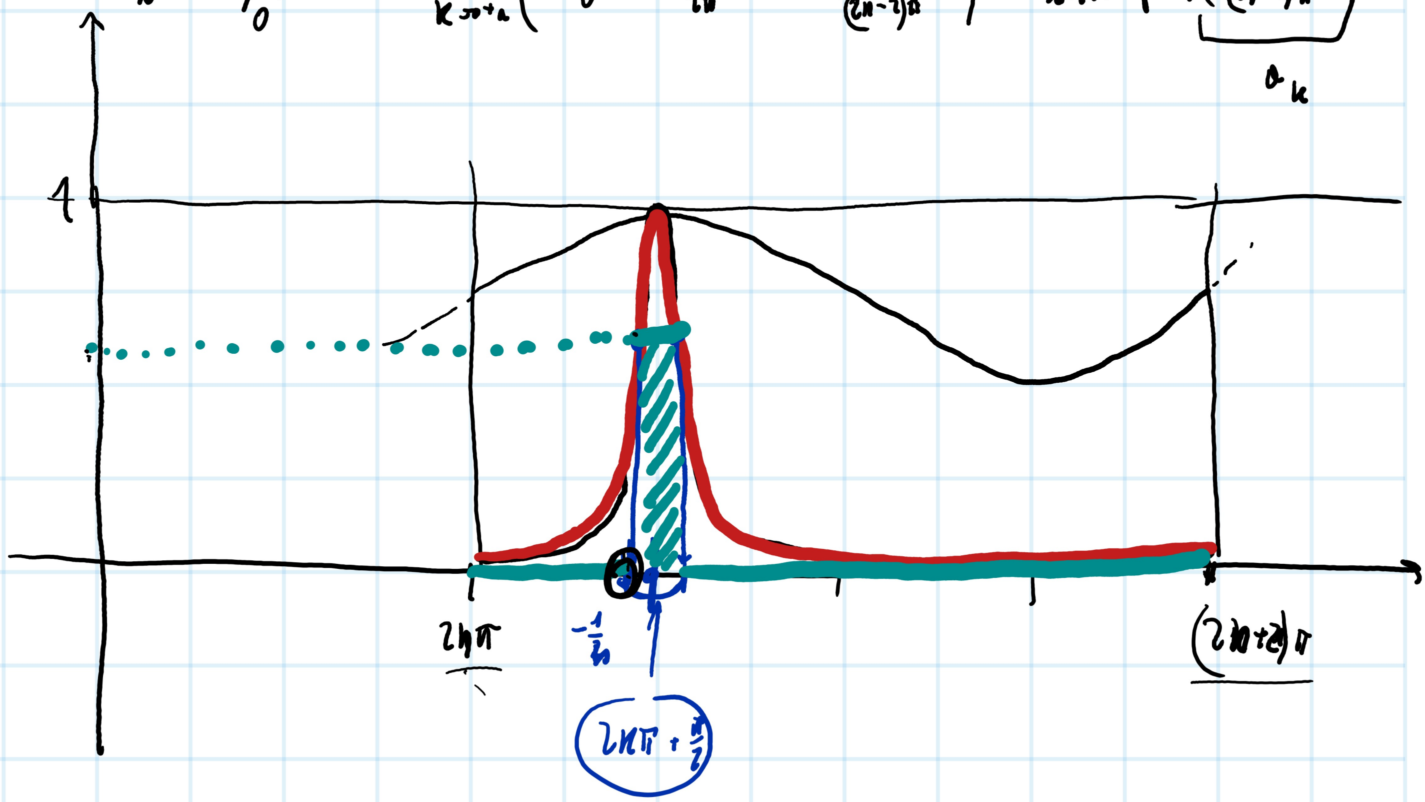
$$\lim_{k\to+\infty}\int_0^{2k\pi} f(x)\,dx = \lim_{k\to+\infty} k\pi = +\infty$$



$b_k = 2k\pi$

$$\int_0^{+\infty} \left(\frac{3}{4} + \frac{\sin x}{4}\right)^x dx$$

$$\lim_{k\to+\infty} \int_0^{2k\pi} f(x)\,dx = \lim_{k\to+\infty}\left(\int_0^{2\pi} + \int_{2\pi}^{4\pi} + \dots + \int_{(2k-2)\pi}^{2k\pi}\right) = \lim_{k\to+\infty} \sum_{p=1}^{k} \underbrace{\left(\int_{(2p-2)\pi}^{2p\pi} f(x)\,dx\right)}_{a_k}$$



$$f\left(2n\pi + \frac{\pi}{2} - \frac{1}{n}\right) \geq \left(\frac{3}{4} + \frac{\sin\left(\cancel{2k\pi} + \frac{\pi}{2} - \frac{1}{n}\right)}{4}\right)^{(2k+2)\pi} = \left(\frac{3 + \cos\frac{1}{n}}{4}\right)$$

$$\underbrace{\left(1 - \frac{1}{2n^2} + O\left(\frac{1}{n^4}\right)\right)^{(2n+2)\pi}}_{\to 1} \left(\frac{3 + 1 - \frac{1}{2n^2} + O\left(\frac{1}{n^4}\right)}{4}\right)^{(2n+2)\pi}$$