Esercitazioni di Analisi Matematica 2 - 10 Aprile 2024 - docente: Callegari

Exe 6: Funzioni integrali - Serie (I)

STUDIARE LE SEGUENTI FUNZIONI INTEGRALI:

$\boxed{1}$ $F(x)=\int_2^x \sqrt{1-\frac{2}{t^3+t^2}}\,dt$

$\boxed{2}$ $F(x)=\int_2^x \frac{e^{\frac{1}{t}}}{t\sqrt{t^2+t-1}}\,dt$

$\boxed{3}$ $F(x)=\int_0^x \arctan\left(e^{\frac{1}{1+t^2}}\right)dt$

STUDIARE IL CARATTERE DELLE SEGUENTI SERIE:

$\boxed{4}$ $\sum \frac{1}{\ln(n!)}$

$\boxed{5}$ $\sum \frac{1}{(\ln n)^{\ln(\ln n)}}$

$\boxed{6}$ $\sum \frac{n!\,n^{n+1}}{(2n)!}A^n$ $(A>0)$

$\boxed{7}$ $\sum \frac{n^{2n}}{(2n)!}A^n$

SOLUZIONI

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$$F(x)=\int_0^x \arctan\left(e^{\frac{1}{1+t^2}}\right)dt \qquad \text{DOMINIO: } \mathbb{R}$$

$$\lim_{x\to+\infty} \int_0^x \underbrace{\arctan\left(e^{\frac{1}{1+t^2}}\right)}_{f(t)}dt = +\infty \qquad \lim_{t\to+\infty} f(t) = \frac{\pi}{4}$$

$$f(t)\geq\frac{\pi}{4} \quad \forall t$$

$$\int_0^{+\infty} f(t)\,dt \quad \text{DIVERGE}$$

DISPARI

$$F(-x)=\int_0^{-x} f(t)\,dt \overset{u=-t}{=} \int_0^x f(-u)(-1)\,du = -\int_0^x f(u)\,du = -F(x)$$

$$m=\lim_{x\to+\infty}\frac{F(x)}{x} \overset{H}{=} \lim_{x\to+\infty}\frac{F'(x)}{1} = \lim_{x\to+\infty}\arctan\left(e^{\frac{1}{1+x^2}}\right) = \frac{\pi}{4}$$

$$q=\lim_{x\to+\infty}\left(F(x)-\frac{\pi}{4}x\right) = \lim_{x\to+\infty}\left(\int_0^x \arctan\left(e^{\frac{1}{1+t^2}}\right)dt - \int_0^x\frac{\pi}{4}dt\right) =$$

$$y=\frac{\pi}{4}x+q \qquad = \lim_{x\to+\infty}\int_0^x\left(\arctan\left(e^{\frac{1}{1+t^2}}\right)-\arctan(1)\right)dt = \text{ESISTE FINITO}$$

$$\int_0^{+\infty}\left(\arctan\left(e^{\frac{1}{1+t^2}}\right)-\arctan(1)\right)dt \quad \text{CONVERGE?}$$

$$0\leq|\bullet|\overset{L}{\leq}\left|e^{\frac{1}{1+t^2}}-1\right| \approx \boxed{\frac{1}{1+t^2}}$$

$\int \bullet\, dt$ converge

$\Downarrow$

$\int \bullet$ converg

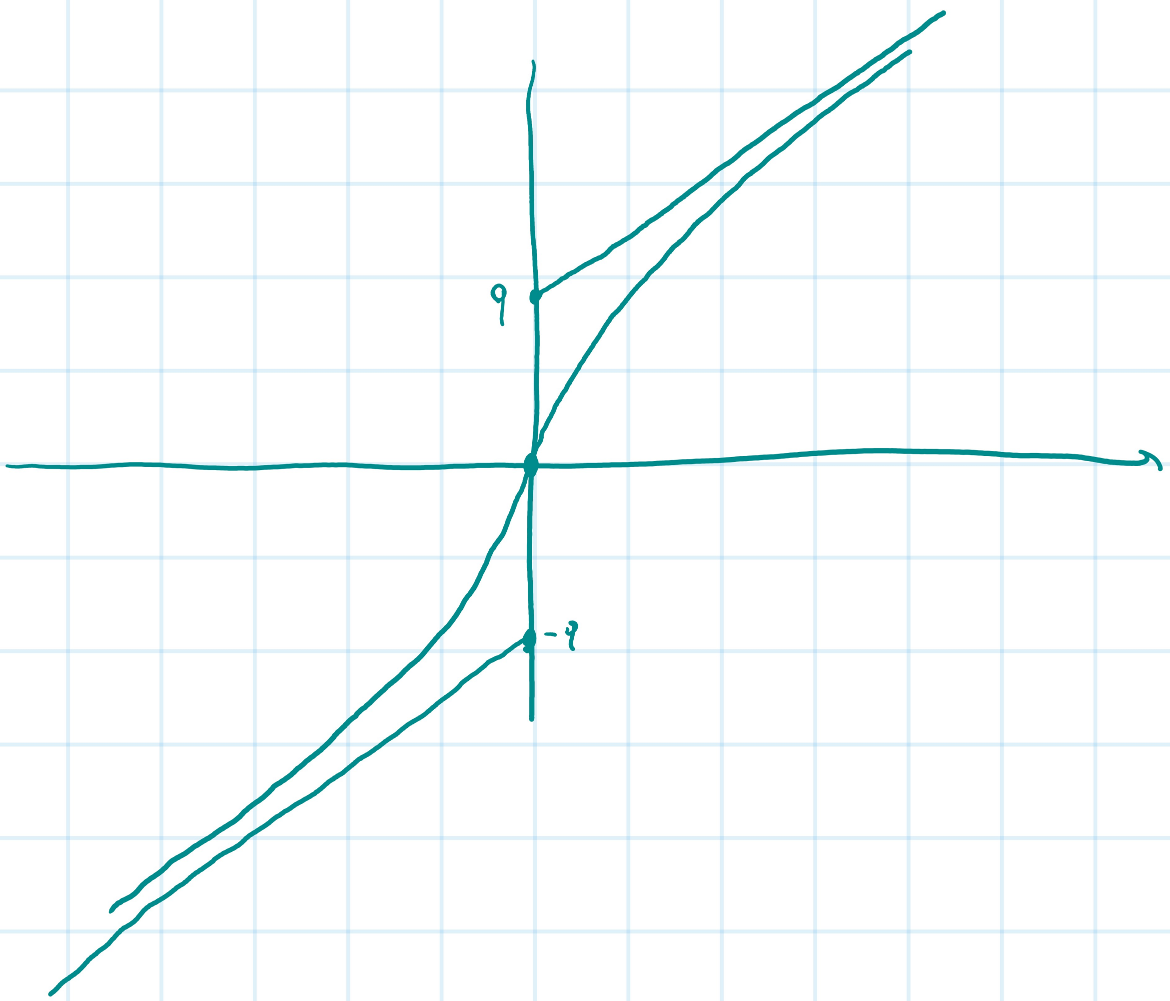
$F'(x) = \underbrace{\arctan\left(e^{\boxed{\frac{1}{1+x^2}}}\right)}$

$\frac{1}{1+x^2}$ cresc. per $x<0$

decr. per $x>0$

$F(x)$ È CONVESSA PER $x<0$

CONCAVA $x>0$



$\boxed{\sum_{n=2}^{+\infty} \frac{1}{\ln(n!)}}$

$\frac{1}{\ln(n!)} > \frac{1}{\ln(n^n)} = \frac{1}{n \ln n}$

$\sum \frac{1}{n \ln n}$

$\sum_{n=n_0}^{+\infty} f(n)$

DIVERGE PER CONFRONTO CON $\int_a^{+\infty} f(x)\,dx$

$\int \frac{1}{x \ln x}$

5 $\sum_{n=10}^{+\infty} \frac{1}{(\ln n)^{\ln(\ln n)}}$

$\ln(\ln n) \quad \sqrt{\ln n}$

$(\ln(n))^{\ln(\ln n)} = e^{\ln(\;)} =$

$= e^{\ln(\ln n)\cdot\ln(\ln n)} < e^{\sqrt{\ln n}\cdot\sqrt{\ln n}} = e^{\ln n} = n$

$\boxed{\frac{1}{(\ln n)^{\ln(\ln n)}} > \frac{1}{n}}$

$\sum \frac{1}{n}$ DIVERGE

$\sum$ DIVERGE

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6 $\sum_{n=2}^{+\infty} \underbrace{\frac{n!\cdot n^{n+1}}{(2n)!}\cdot A^n}_{a_n} \qquad (A>0)$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!\cdot(n+1)^{n+2}}{\cancel{(2n+2)!}}\cdot A^{\cancel{n+1}}\cdot\frac{\cancel{(2n)!}}{\cancel{n!}\cdot n^{n+1}}\cdot\frac{1}{\cancel{A^n}} =$$

$(2n+2)(2n+1)$

$$= \frac{(n+1)^2}{(2n+2)(2n+1)}\cdot\boxed{\frac{(n+1)^{n+1}}{n^{n+1}}}\cdot A = \underbrace{\frac{A}{4}}_{\to \frac{A}{4}}\cdot\underbrace{\frac{(n+1)^{2}}{\cancel{(n+1)}(n+\frac{1}{2})}}_{\to 1}\cdot\underbrace{\left(1+\frac{1}{n}\right)^{n+1}}_{e} \to \frac{A}{4}\cdot e$$

$\left(\frac{n+1}{n}\right)$

$$\begin{cases} <1 & A<\frac{4}{e} \\ 1 & A=\frac{4}{e} \\ >1 & A>\frac{4}{e} \end{cases}$$

$A > \frac{4}{e}$ DIVERGE $\qquad A = \frac{4}{e}$ (??) DIVERGE

$A < \frac{4}{e}$ CONVERGE

$\frac{a_{n+1}}{a_n} \leq \ell < 1 \Rightarrow$ CONVERGE

$$\boxed{\frac{a_{n+1}}{a_n} \geq 1 \Rightarrow \text{DIVERGE}}$$

$a_{n+1} \geq a_n$

$A = \frac{4}{e}$

$\frac{2n+1+1}{2n+1}$ $\quad \frac{2n+2}{2n+1}$

$$\frac{a_{n+1}}{a_n} = \frac{1}{e} \cdot \frac{n+1}{n+\frac{1}{2}} \cdot \left(1+\frac{1}{n}\right)^{n+1} =$$

$$= \left(1+\frac{1}{2n+1}\right) \cdot \frac{1}{e} \cdot \left(1+\frac{1}{n}\right)^{n+1} \to 1 \text{ DECR.}$$

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$$\sum_{n=1}^{+\infty} \underbrace{\frac{n^{2n}}{(2n)!} A^n}_{a_n} \qquad (A>0)$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{2n+2}}{\underbrace{\cancel{(2n+2)!}}_{(2n+2)(2n+1)}} \cdot A^{n+1} \cdot \frac{\cancel{(2n)!}}{n^{2n}\, A^n} =$$

$$= \frac{(n+1)^2}{2\cancel{(n+1)}(2n+1)} \cdot A \cdot \frac{(n+1)^{2n}}{n^{2n}} =$$

$$= \underbrace{\frac{A}{4}}_{\downarrow\ \frac{A}{4}} \cdot \underbrace{\frac{2n+2}{2n+1}}_{\downarrow\ 1} \cdot \underbrace{\left(1+\frac{1}{n}\right)^{2n}}_{\downarrow\ e^2} \longrightarrow A\cdot\frac{e^2}{4}$$

$$\Downarrow$$

$$\begin{cases} A < \frac{4}{e^2} & \text{CONV.} \\ A > \frac{4}{e^2} & \text{DIV} \\ A = \frac{4}{e^2} & \end{cases}$$

$A = \frac{4}{e^2}$

$$\boxed{\frac{a_{n+1}}{a_n}} = \frac{2n+2}{2n+1} \cdot \left(1+\frac{1}{n}\right)^{2n} \cdot \frac{1}{e^2} =$$

$$= \left(1+\frac{1}{2n+1}\right) \cdot e^{2n\ln\left(1+\frac{1}{n}\right)-2} =$$

$$2n\ln\left(1+\frac{1}{n}\right)-2 =$$

$$= 2n\cdot\left(\cancel{\frac{1}{n}} - \frac{1}{2n^2} + \frac{1}{3n^3} + o\left(\frac{1}{n^4}\right)\right) \cancel{-2} =$$

$$= \boxed{-\frac{1}{n} + \frac{2}{3n^2} + o\left(\frac{1}{n^3}\right)}$$

$$= \left(1+\frac{1}{2n+1}\right)\left(1+\square+o\left(\square^2\right)\right)$$

$$e^{\square} = 1+\square+o(\square^2)$$

$$= \left(1+\frac{1}{2n+1}\right)\cdot\left(1-\frac{1}{n}+\underbrace{\frac{2}{3n^2}}+\underbrace{o\left(\frac{1}{n^3}\right)}+\underbrace{o\left(\frac{1}{n^2}\right)}\right) =$$

$$= \left(1+\frac{1}{2n+1}\right)\cdot\left(1-\frac{1}{n}+o\left(\frac{1}{n^2}\right)\right) =$$

$$= \left(1+\frac{1}{2n}+o\left(\frac{1}{n^2}\right)\right)\left(1-\frac{1}{n}+o\left(\frac{1}{n^2}\right)\right) = 1+\frac{1}{2n}-\frac{1}{n}+o\left(\frac{1}{n^2}\right) = \boxed{1-\frac{1}{2n}+o\left(\frac{1}{n^2}\right)}$$

$$\left(1+\frac{1}{2n+1}\right) = 1+\left(\frac{1}{2n+1}-\frac{1}{2n}\right)+\frac{1}{2n} = 1+\frac{1}{2n}+\frac{2n-(2n+1)}{(2n+1)\cdot 2n} =$$

$$= 1+\frac{1}{2n}-\underbrace{\boxed{\frac{1}{(2n+1)(2n)}}}_{o\left(\frac{1}{n^2}\right)}$$

$b_n = \frac{1}{n}$ $\quad \sum b_n$ DIVERGE

$$\boxed{\frac{b_{n+1}}{b_n}} = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} = \frac{n+1-1}{n+1} = 1 - \frac{1}{n+1} = 1 - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n} + \frac{\cancel{n}+1-\cancel{n}}{n(n+1)} = 1 - \frac{1}{n} + \frac{1}{n(n+1)} =$$

$$= \boxed{1 - \frac{1}{n} + O\left(\frac{1}{n^2}\right)}$$

$$\boxed{\frac{a_{n+1}}{a_n}} = \ldots\ldots = \boxed{1 - \frac{1}{2n} + O\left(\frac{1}{n^2}\right)}$$

DEF. IN n $\quad \frac{a_{n+1}}{a_n} > \frac{b_{n+1}}{b_n}$

$n \geq n_0$

$$\frac{a_{n_0+1}}{a_{n_0}} > \frac{b_{n_0+1}}{b_{n_0}} \qquad n_0$$

$$\frac{a_{n_0+2}}{a_{n_0+1}} > \frac{b_{n_0+2}}{b_{n_0+1}} \qquad n_0+1$$

$$\vdots$$

$$\frac{a_n}{a_{n-1}} > \frac{b_n}{b_{n-1}} \qquad n-1$$

$$\frac{\cancel{a_{n_0+1}}}{a_{n_0}} \cdot \frac{\cancel{a_{n_0+2}}}{\cancel{a_{n_0+1}}} \cdots \frac{a_n}{\cancel{a_{n-1}}} > \frac{\cancel{b_{n_0+1}}}{b_{n_0}} \cdot \frac{\cancel{b_{n_0+2}}}{\cancel{b_{n_0+1}}} \cdots \frac{b_n}{\cancel{b_{n-1}}}$$

$$\boxed{\frac{a_n}{a_{n_0}} > \frac{b_n}{b_{n_0}}}$$

$$a_n > \boxed{\frac{a_{n_0}}{b_{n_0}}} \cdot b_n$$

$$\sum a_n$$

$$\frac{a_{n+1}}{a_n} = \underbrace{1 - \frac{\lambda}{n} + O\left(\frac{1}{n^2}\right)}$$

$$> 1 - \frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

$\lambda < 1$ $\boxed{\text{DIV}}$

$\lambda > 1$ $\boxed{\text{CONV.}}$

$$b_n = \frac{1}{n^\alpha} \qquad \frac{b_{n+1}}{b_n} = \frac{\frac{1}{(n+1)^\alpha}}{\frac{1}{n^\alpha}} = \left(\frac{n}{n+1}\right)^\alpha = \left(\frac{n+1-1}{n+1}\right)^\alpha =$$

$$= \left(1 - \frac{1}{n+1}\right)^\alpha = 1 - \frac{\alpha}{n+1} + \underline{O\left(\frac{1}{(n+1)^2}\right)}$$

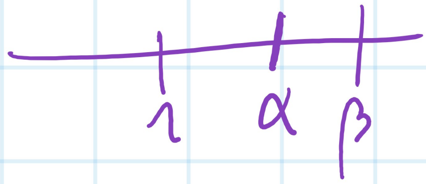
$$= 1 - \alpha\cdot\left(\frac{1}{n} - \frac{1}{n} + \frac{1}{n+1}\right) + O\left(\frac{1}{n^2}\right) =$$

$$= \boxed{1 - \frac{\alpha}{n} + O\left(\frac{1}{n^2}\right)} = \frac{b_{n+1}}{b_n}$$

$$= -\frac{1}{n(n+1)} = O\left(\frac{1}{n^2}\right)$$

$\boxed{\beta > 1}$

$$\boxed{\frac{a_{n+1}}{a_n} = 1 - \frac{\beta}{n} + O\left(\frac{1}{n^2}\right)}$$



$$\sum \boxed{\frac{1}{n^\alpha}} \qquad \alpha > 1$$

Converge.

$$\alpha < \beta \Rightarrow 1 - \frac{\beta}{n} < 1 - \frac{\alpha}{n}$$

$\sum b_n$ converge

$\Downarrow$

$$\frac{a_{n+1}}{a_n} \qquad 1 - \frac{\beta}{n} + O\left(\frac{1}{n^2}\right) < 1 - \frac{\alpha}{n} + O\left(\frac{1}{n^2}\right) \qquad \frac{b_{n+1}}{b_n}$$

$\Downarrow$

$\sum a_n$ Converge.