Exe 7: Serie (II)

STUDIARE IL CARATTERE DI $\sum a_n$ NEI SEGUENTI CASI

$\boxed{1}$ $\begin{cases} a_{n+1} = \sqrt{1+a_n} - 1 \\ a_0 = 1 \end{cases}$

$\boxed{2}$ $a_n = \int_n^{n+1} \sin(2\pi x)\, dx$

$\boxed{3}$ $a_n = \int_n^{n+1} \sin x \, dx$

STUDIARE IL CARATTERE DELLE SEGUENTI SERIE

$\boxed{4}$ $\sum \arctan \frac{(-1)^n}{\sqrt{n}}$

$\boxed{5}$ $\sum \left(e^{\frac{(-1)^n}{\sqrt{n}}} - 1 \right)$

$\boxed{6}$ $\sum \frac{\cos(\pi n)}{\sqrt{n} + (1+(-1)^n) n^2}$

$\boxed{7}$ $\sum \frac{(-1)^n}{\sqrt{n} + (-1)^n}$

$\boxed{8}$ $\sum \frac{(-1)^{[\ln n]}}{n}$

$\boxed{9}$ $\sum \frac{\cos(\frac{\pi}{2} n)}{n + \sin n}$

$\boxed{10}$ $\sum \frac{\cos n}{n^\alpha} \quad (\alpha > 0)$

$\boxed{11}$ $\sum \frac{\sin n}{n + 10 \sin n}$

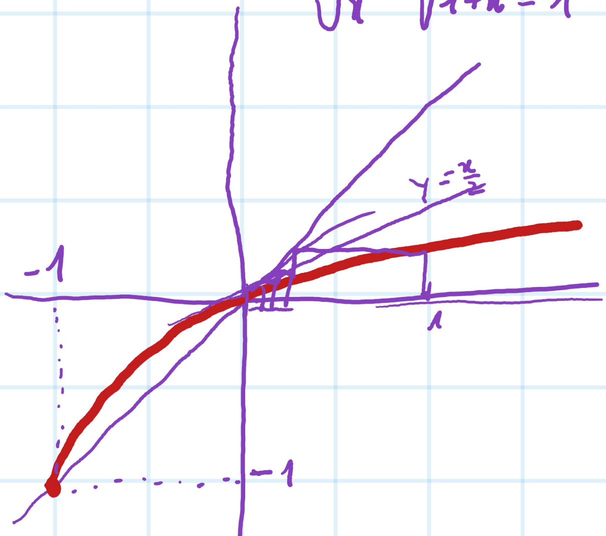
SOLUZIONI

$f(x) = \sqrt{1+x} - 1$

$\sqrt{x} \qquad \sqrt{1+x} - 1$

1

$$\begin{cases} a_{n+1} = \sqrt{1+a_n} - 1 \\ a_0 = 1 \end{cases} \qquad \sum a_n$$



$\boxed{a_n \text{ DECRESC}}$ $\begin{cases} 1)\ f(x) \le x \\ 2)\ f \text{ È CRESC.} \end{cases}$

$$\boxed{\frac{a_{n+1}}{a_n}} = \frac{\sqrt{1+a_n} - 1}{a_n} \to \frac{1}{2} \qquad \frac{\sqrt{1+x} - 1}{x} \xrightarrow[x \to 0]{} \frac{1}{2}$$

4

$$\sum_{n=1}^{+\infty} \arctan \frac{(-1)^n}{\sqrt{n}}$$

NO

$$\arctan \frac{(-1)^n}{\sqrt{n}} \underset{\text{CONF. ASINT.}}{\approx} \boxed{\frac{(-1)^n}{\sqrt{n}}} \qquad n \to +\infty$$

$$(-1)^n \cdot \boxed{\frac{1}{\sqrt{n}}} \quad \boxed{\text{converge}} \qquad \frac{1}{\sqrt{n}} \downarrow 0 \text{ DECR.}$$

SI

$$\arctan \frac{(-1)^n}{\sqrt{n}} = \underbrace{(-1)^n}_{\text{SEGNO ALTERNO}} \underbrace{\arctan \frac{1}{\sqrt{n}}}_{\to 0 \text{ DECRES.}} \qquad \text{CONV. PER } L.$$

$\boxed{5}$ $$\sum_{n=1}^{+\infty}\left(e^{\frac{(-1)^n}{\sqrt{n}}}-1\right)$$

$$\underline{e^{\frac{(-1)^n}{\sqrt{n}}}-1} \simeq \underline{\frac{(-1)^n}{\sqrt{n}}} \qquad \boxed{e^x-1\simeq x}$$

$$e^x-1=x+\frac{1}{2}x^2+o(x^2)$$

$$e^{\frac{(-1)^n}{\sqrt{n}}}-1=\frac{(-1)^n}{\sqrt{n}}+\frac{1}{2}\left(\frac{(-1)^n}{\sqrt{n}}\right)^2+o\left(\left(\frac{(-1)^n}{\sqrt{n}}\right)^3\right)=$$

$$=\boxed{\frac{(-1)^n}{\sqrt{n}}}+\boxed{\frac{1}{2}\cdot\frac{1}{n}+o\left(\frac{1}{n}\right)}$$

$$\sum_{n=1}^{+\infty}\left(e^{\frac{(-1)^n}{\sqrt{n}}}-1-\frac{(-1)^n}{\sqrt{n}}\right)=\sum_{n=1}^{+\infty}\left(\frac{1}{2n}+o\left(\frac{1}{n}\right)\right)$$

NON CAMBIA CARATTERE

PERCHÉ SUA SERIE CONVERGE

$$\boxed{\frac{1}{2n}+o\left(\frac{1}{n}\right)\simeq\frac{1}{2n}} \qquad \boxed{\sum\frac{1}{2n} \text{ DIVERGE}}$$

SEGNO COSTANTE

$$\frac{1}{2n}\cdot\left(\underbrace{1+o\left(\overset{\downarrow}{1}\right)}\right)$$

$$\sum\left(\frac{1}{2n}+o\left(\frac{1}{n}\right)\right) \text{ DIVERGE PER}$$

CONF. ASINTOTICO

CON

$\boxed{6}$ $\sum \frac{\overset{(-1)^n}{\cos(\pi n)}}{\sqrt{n}+(1+(-1)^n)\,n^2}$

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{\sqrt{n}+(1+(-1)^n)n^2}$$

$$\frac{1}{\sqrt{n}+(1+(-1)^n)\cdot n^2} = \begin{cases} \frac{1}{\sqrt{n}} & n \text{ DISPARI} \\ \frac{1}{\sqrt{n}+2n^2} & n \text{ PARI} \end{cases}$$

$S_n \to -\infty$

$\shortparallel$

$\sum_{n=1}^{n} a_n$

$S_{2n} \to -\infty$

S_{2n+1}

$a_{100} = \frac{1}{\sqrt{100}+2\cdot 100^2}$ $\quad a_{100} < a_{101}$

$a_{101} = \frac{1}{\sqrt{101}}$

$S_{2n+1} - S_{2n} = a_{2n+1}$

$$\sum_{n=1}^{+\infty} (-1)^n$$

$S_{2n} = 0 \to 0$ $(-1+1)(-1+1)-1+1)$

$$S_{2n} = \boxed{\sum_{k=1}^{2n} a_k} = \boxed{\sum_{k=0}^{n-1} \left(a_{2k+1} + a_{2k+2}\right)} =$$

$$= \sum_{k=0}^{n-1} \left(\frac{(-1)^{2k+1}}{\sqrt{2k+1} + \left(1+(-1)^{2k+1}\right)\cdot(2k+1)^2} + \frac{(-1)^{2k+2}}{\sqrt{2k+2} + \left(1+(-1)^{2k+2}\right)(2k+2)^2} \right) =$$

$$= \sum_{k=0}^{n-1} \left(-\frac{1}{\sqrt{2k+1}} + \boxed{\frac{1}{\sqrt{2k+2} + (2k+2)^2}} \right) \leq \sum_{k=0}^{n-1} \left(-\frac{1}{2\sqrt{2k+1}} \right) \leq$$

$$< -\frac{1}{2}\frac{1}{\sqrt{2k+1}}$$

1

$$2k+1 < k^2 \qquad \frac{1}{2\sqrt{2k+1}} > \frac{1}{2\sqrt{k^2}} = \frac{1}{2k}$$

$$-\frac{1}{2\sqrt{2k+1}} < -\frac{1}{2k}$$

$$\leq \sum_{k=1}^{n-1} -\frac{1}{2k} = -\frac{1}{2} \boxed{\sum_{k=1}^{n-1} \frac{1}{k}} \longrightarrow -\infty \quad \text{PER } n \to +\infty$$

PERCHÉ È S_n DI SERIE ARMONICA

$$\overbrace{S_{2n+1}}^{-\infty} = \underbrace{S_{2n}}_{-\infty} + \underbrace{a_{2n+1}}_{0}$$

$\sum a_n$ HA STESSO CARATTER

$\sum (a_n - b_n)$

$$a_n = \begin{cases} \cancel{\dfrac{1}{\sqrt{n}+2n^2}} & \text{PARI} \\ -\dfrac{1}{\sqrt{n}} & \text{DISPAR} \end{cases}$$

$$b_n = \begin{cases} \dfrac{1}{\sqrt{n}+2n^2} & \text{PARI} \\ 0 & \text{DISPARI} \end{cases}$$

$\boxed{\sum b_n \text{ converge PER CONFRONTO CON } \sum c_n}$

$$\boxed{c_n = \frac{1}{\sqrt{n}+2n^2}} \sim \frac{1}{2} \cdot \frac{1}{n^2}$$

$\sum c_n$ converge

n^{α} $(\alpha > 0)$

$$\boxed{7} \quad \sum_{n=3}^{+\infty} \frac{(-1)^n}{\sqrt{n} + (-1)^n} = \sum_{n=3}^{+\infty} (-1)^n \cdot \frac{1}{\sqrt{n}+(-1)^n}$$

$$S_{2n} = \sum_{k=3}^{2n} a_k = \sum_{k=1}^{n-1} \left(a_{2k+1} + a_{2k+2} \right) =$$

$$= \sum_{k=1}^{n-1} \left(\frac{(-1)^{2k+1}}{\sqrt{2k+1} + (-1)^{2k+1}} + \frac{(-1)^{2k+2}}{\sqrt{2k+2} + (-1)^{2k+2}} \right)$$

$$= \sum_{k=1}^{n-1} \left(- \frac{1}{\sqrt{2k+1} - 1} + \frac{1}{\sqrt{2k+2} + 1} \right)$$

$$= \sum_{k=1}^{n-1} \boxed{\frac{-\sqrt{2k+2} - 1 + \sqrt{2k+1} - 1}{\boxed{(\sqrt{2k+1} - 1)(\sqrt{2k+2} + 1)}}} =$$

$$= \frac{-3}{2k}$$

$$\frac{-3}{(\sqrt{2k+2} - 1)(\sqrt{2k+1}+1)} <$$

$$< \frac{-3}{\boxed{}} < \bigcirc < \frac{-2}{\boxed{}} < \frac{-2}{(\sqrt{2k+2} - 1)(\sqrt{2k+2} + 1)} =$$

$$= - \frac{2}{2k+1}$$

$$\boxed{-\frac{3}{2k} < \boxed{} < -\frac{2}{2k+1}}$$

$$S_{2n} < \boxed{\sum_{k=1}^{n-1} -\frac{2}{2k+1}} = -\boxed{\sum_{k=1}^{n-1} \frac{2}{2k+1}} \longrightarrow -\infty$$

$$\downarrow$$

$$-\infty$$

$$\sum_{n=1}^{+\infty} \frac{2}{2n+1} \quad \text{DIVERGE}$$

$\boxed{8}$ $$\sum_{n=1}^{+\infty} \frac{(-1)^{\lfloor \log_2 n \rfloor}}{n}$$

n	1	2	3	4	5	6	7	8	9	10	…	15
$\lfloor \log_2 n \rfloor$	0	1	1	2	2	2	2	3	3	3	…	3

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{1}{8} - \frac{1}{9} - \dots - \frac{1}{15}$$

$$\left| \sum_{k=2^p}^{2^{p+1}-1} \frac{(-1)^{\lfloor \log_2 k \rfloor}}{k} \right| = \left| \sum_{k=2^p}^{2^{p+1}-1} \frac{(-1)^p}{k} \right| =$$

$$= \left| (-1)^p \sum_{k=2^p}^{2^{p+1}-1} \frac{1}{k} \right| = \sum_{k=2^p}^{2^{p+1}-1} \frac{1}{k} \geq \sum_{k=2^p}^{2^{p+1}-1} \frac{1}{2^{p+1}} = \frac{1}{2^{p+1}} \sum_{k=2^p}^{2^{p+1}-1} 1 =$$

$$= \frac{1}{2^{p+1}} \cdot 2^p = \frac{1}{2}$$

$\boxed{11}$ $\sum \dfrac{\sin n}{n + 10 \sin n}$

$\sum \dfrac{\sin n}{n}$ Converge

$$\sum \left(\frac{\sin n}{n + 10 \sin n} - \frac{\sin n}{n} \right) =$$

$$= \sum \sin n \cdot \frac{\cancel{n} - \cancel{n} - 10 \sin n}{(n + 10 \sin n) \cdot n} =$$

$$= \sum \boxed{\frac{-10 \sin^2 n}{(n + 10 \sin n) \cdot n}}$$

$n \geq 20 \Rightarrow \boxed{n - 10 \geq \frac{n}{2}}$

$$|\bullet| \leq \frac{10}{(n-10) n} \leq \frac{10}{\frac{n}{2} \cdot n} = \boxed{\frac{20}{n^2}}$$

$\sum \bullet$ Converge

$\sum |\bullet|$ Converge

$\Downarrow$

Crit. ass. Converg. $\rightarrow$

$\sum \bullet$ Converge