NUMERI COMPLESSI (BOZZA)

INDICE PROVVISORIO:

ESERCIZI SUI NUMERI COMPLESSI

CALCOLARE LE SEGUENTI ESPRESSIONI:

1 $1 + i + (i)^2 + (i)^3 + (i)^4 + \dots + i^{50}$

2 $(1+i)^{10}$.

3 $\dfrac{(i-\sqrt{3})^{69}}{(1+i)^{139}}$

4 $\left((\sqrt{3}-1) + i(\sqrt{3}+1)\right)^{12}$

TROVARE TUTTE LE RADICI n-ESIME DI z (IN FORMA CARTESIANA) NEI SEGUENTI CASI:

5 $z = 1 \qquad n = 3, 4, 6, 8$

6 $z = -1 \qquad n = 2, 3, 4, 8$

7 $z = i \qquad n = 2, 3, 4$

8 $z = -1+i \qquad n = 2, 3$

9 $z = \dfrac{(i-\sqrt{3})^{16}}{(1+i)^{28}} \qquad n = 4$

10 $z = \dfrac{(5+12i)^4}{(7-24i)^2} \qquad n = 8$

CALCOLARE

11 LA SOMMA E IL PRODOTTO DI TUTTE LE RADICI CUBICHE DI $1+i$

12 LA SOMMA E IL PRODOTTO DI TUTTE LE RADICI 16^e DI $1+i$

13 LA SOMMA DEI QUADRATI DI TUTTE LE RADICI OTTAVE DI $-8-8\sqrt{3}i$ AVENTI PARTE IMMAGINARIA NEGATIVA.

14 IL PRODOTTO DI TUTTE LE RADICI 16^e DI $3+4i$ CON PARTE IMMAGINARIA POSITIVA.

RISPONDERE AI SEGUENTI QUESITI:

15 DATO $z_0 = \left(\frac{500 + 250i}{11 + 2i} \right)^4$ TROVARNE IN FORMA CARTESIANA:

a) TUTTE LE RADICI QUARTE

b) TUTTE LE RADICI OTTAVE

c) LA SOMMA DI TUTTE LE RADICI SEDICESIME

16 DATO $z_0 = \frac{5^{12} \cdot (3+4i)^{36}}{(2-i)^{12} \cdot (2+11i)^{24}}$ TROVARNE IN FORMA CARTESIANA:

a) UNA RADICE DODICESIMA

b) TUTTE LE RADICI QUARTE

17 DATI $z_1 = -\sqrt{3} + i$ E $z_2 = \sqrt{2} + i\sqrt{2}$,

a) TROVARE IN FORMA CARTESIANA z_1^{96} E z_2^{96};

b) PRESI $n \in \mathbb{N} - \{0\}$ E $z \in \mathbb{C}$ TALI CHE SIA z_1 CHE z_2 SIANO RADICI n-ESIME DI z, DIRE QUAL È IL MINIMO VALORE POSSIBILE PER $|z|$.

18 DATI $z_1 = \sqrt{2+\sqrt{3}} + i\sqrt{2-\sqrt{3}}$ E $z_2 = \sqrt{2-\sqrt{2}} - i\sqrt{2+\sqrt{2}}$,

a) TROVARE z_1^{2022} E, IN PARTICOLARE, $\lfloor \operatorname{Im} z_1^{2022} \rfloor$.

b) DETERMINARE L'INSIEME $I = \{ n \in \mathbb{Z} \mid z_1^n = z_2^n \}$

19 DOPO AVER RISOLTO L'EQUAZIONE $z^{18} = z^{10}$ TROVARE TUTTI I NUMERI COMPLESSI PER I QUALI L'INSIEME DELLE RADICI DICIOTTESIME NON È DISGIUNTO DA QUELLO DELLE RADICI DECIME.

RISOLVERE IN $\mathbb{C}$ LE SEGUENTI EQUAZIONI:

20 $z^4 = z^2 + 2$

21 $z^4 = |z|^2 + 2$

22 $z^{10} \cdot \overline{z}^8 = 512 i$

23 $|z| \cdot \left((1+i) \cdot z \right)^2 = 16$

24 $|z| = z^2 + z - 12 + \overline{z}$

25 $\left(\frac{z}{3+4i} \right)^6 = \left(\frac{\overline{z}}{3-4i} \right)^2$

26 $(2z)^8 = -\frac{(i+\sqrt{3})^{16} \cdot (1+i)^{12}}{64}$

27 $\left(z^2 + 2|z| + 1 \right) \cdot \left(3 + \frac{z^4}{|\overline{z}|^4 - 1} \right) = 0$

DESCRIVERE/DISEGNARE NEL PIANO COMPLESSO I SEGUENTI INSIEMI:

28 $A = \{ z \in \mathbb{C} \mid |z+i| + |z-i| = 4 \}$

29 $A = \{ z \in \mathbb{C} \mid \arg\left(\frac{z-1}{z+1} \right) = \frac{\pi}{4} \}$

30 $A = \{ z \in \mathbb{C} \mid |z+1| = 2|z| \}$

31 $A = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 ,\ \operatorname{Im}(z^2) > 0 ,\ \operatorname{Im}(z^3) < 0 \}$

32 $A = \{ z \in \mathbb{C} \mid z \cdot \overline{z} < 2 ,\ \operatorname{Im}\left(\frac{\overline{z}}{z} \right) = 1 \}$

33 $A = \{ z \in \mathbb{C} \mid \operatorname{Re}(z) < \operatorname{Re}(z^2) \}$

RISPONDERE AI SEGUENTI QUESITI:

34 DEL NUMERO COMPLESSO z SAPPIAMO SOLO CHE $z^{24} = 17 + i$.
QUANTI DIVERSI VALORI PUÒ ASSUMERE z^{15}?

35 QUANTI SONO GLI ELEMENTI DELL'INSIEME $\left\{ z \in \mathbb{C} \,\middle|\, |z| \leq 25,\ e^{z} = \frac{24+7i}{25e^{20}} \right\}$?

36 DATI $z_1, z_2, z_3 \in \mathbb{C}$ TALI CHE IL TRIANGOLO DI CUI SONO VERTICI SIA EQUILATERO,
CHE VALORI PUÒ ASSUMERE $\left(\frac{z_1 - z_2}{z_1 - z_3} \right)^3$?

37 DEL POLINOMIO $p(z)$ SAPPIAMO CHE:

a) I COEFFICIENTI SONO TUTTI INTERI,

b) È DIVISIBILE PER $q(z) = z^2 + z + 1$,

c) SE z_0 È SOLUZIONE DI $p(z) = 0$ ALLORA ANCHE iz_0 LO È.

QUAL È IL MINIMO GRADO CHE PUÒ AVERE $p(z)$?

38 TROVARE LA FRONTIERA DELL'INSIEME $A = \left\{ \left(\frac{3}{5} + \frac{4}{5} i \right)^n \,\middle|\, n \in \mathbb{N} \right\}$

DIRE PER QUALI $z \in \mathbb{C}$ CONVERGONO LE SEGUENTI SERIE DI POTENZE:

39 $\displaystyle\sum_{n=1}^{+\infty} \frac{(3n)!}{(n!)^3} z^n$

40 $\displaystyle\sum_{n=1}^{+\infty} \left(1 + \frac{1+i}{n} \right)^{n^2} z^n$

41 $\displaystyle\sum_{n=1}^{+\infty} \frac{(n + \sqrt{n})^n}{n^{n+\sqrt{n}}} z^n$

42 $\displaystyle\sum_{n=1}^{+\infty} \frac{\left(\sqrt{n^4 + n^2} \right)^{n^2}}{(n^2)^{\sqrt{n^4 + n^2}}} \cdot z^n$

RISPOSTE:

1 i

2 $32i$

3 $-\frac{1}{2}+\frac{i}{2}$

4 -2^{18}

5

$n=3$: 1 $\quad -\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

$n=4$: ± 1 $\quad \pm i$

$n=6$: ± 1 $\quad \frac{1}{2}\pm\frac{\sqrt{3}}{2}i$ $\quad -\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

$n=8$: ± 1 $\quad \pm i$ $\quad \frac{1}{\sqrt{2}}\pm\frac{i}{\sqrt{2}}$ $\quad -\frac{1}{\sqrt{2}}\pm\frac{i}{\sqrt{2}}$

6

$n=2$: $\pm i$

$n=3$: -1 $\quad \frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

$n=4$: $\frac{1}{\sqrt{2}}\pm\frac{i}{\sqrt{2}}$ $\quad -\frac{1}{\sqrt{2}}\pm\frac{i}{\sqrt{2}}$

$n=6$: $\frac{\sqrt{3}}{2}\pm\frac{i}{2}$ $\quad \pm i$ $\quad -\frac{\sqrt{3}}{2}\pm\frac{i}{2}$

$n=8$: $A\pm iB$ $\quad -A\pm iB$ $\quad B\pm iA$ $\quad -B\pm iA$ $\quad \left(\text{DOVE } A=\sqrt{\frac{2-\sqrt{2}}{4}} \text{ E } B=\sqrt{\frac{2+\sqrt{2}}{4}}\right)$

7

$n=2$: $\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}$ $\quad -\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}$

$n=3$: $\pm\frac{\sqrt{3}}{2}+\frac{i}{2}$ $\quad \pm\frac{\sqrt{3}}{2}+\frac{i}{3}$

$n=4$: $\pm(B+iA)$ $\quad \pm(A-iB)$ $\quad \left(\text{DOVE } A=\sqrt{\frac{2-\sqrt{2}}{4}} \text{ E } B=\sqrt{\frac{2+\sqrt{2}}{4}}\right)$

8

$n=2$: $\pm\left(\sqrt{\frac{\sqrt{2}-1}{2}}+i\cdot\sqrt{\frac{\sqrt{2}+1}{2}}\right)$

$n=3$: $\frac{1}{\sqrt[3]{2}}+\frac{i}{\sqrt[3]{2}}$ $\quad -\frac{\sqrt{3}+1}{2\sqrt[3]{2}}+\frac{\sqrt{3}-1}{2\sqrt[3]{2}}i$ $\quad \frac{\sqrt{3}-1}{2\sqrt[3]{2}}-\frac{\sqrt{3}+1}{2\sqrt[3]{2}}i$

9 $\pm\left(\frac{\sqrt{3}+1}{2}+\frac{\sqrt{3}-1}{2}i\right)$ $\quad \pm\left(\frac{\sqrt{3}-1}{2}-\frac{\sqrt{3}+1}{2}i\right)$

10 $\pm\left(\frac{8}{5}+\frac{i}{5}\right)$ $\quad \pm\left(\frac{1}{5}-\frac{8}{5}i\right)$ $\quad \pm\left(\frac{7\sqrt{2}}{\sqrt{10}}+\frac{9\sqrt{2}}{\sqrt{10}}i\right)$ $\quad \pm\left(\frac{9\sqrt{2}}{\sqrt{10}}-\frac{7\sqrt{2}}{\sqrt{10}}i\right)$

11 SOMMA $= 0$ $\quad$ PRODOTTO $= 1+i$

12 SOMMA $= 0$ $\quad$ PRODOTTO $= 1+i$

13 0

14 $1-2i$

15
- **a** $\pm(48+14i)$ $\quad \pm(14-48i)$
- **b** $\pm(7+i)$ $\quad \pm(1-7i)$ $\quad \pm(3+4i)$ $\quad \pm(4-3i)$
- **c** 0

16 **a** $2+i$ $\quad$ **b** $\pm(2+11i)$ $\quad \pm(11-2i)$

17 **a** $z_1^{96}=z_2^{96}=2^{96}$ $\quad$ **b** 2^{24}

18 **a** $z_1^{2022}=2^{2022}i$ $\quad$ **b** $I=\{48n \mid n\in\mathbb{Z}\}$

19 $z^{18}=z^{10} \iff z\in\left\{0,1,-1,i,-i,\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}},\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}},-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}},-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right\}$

$\{z\in\mathbb{C} \mid \exists w\in\mathbb{C} \text{ t.c. } w^{10}=z=w^{18}\}=\{0,1,-1,i,-i\}$

20 $\pm i$, $\pm\sqrt{2}$

21 $\sqrt{2}\pm i\sqrt{2}$, $-\sqrt{2}\pm i\sqrt{2}$

22 $1\pm i$

23 $\pm(\sqrt{2}-i\sqrt{2})$

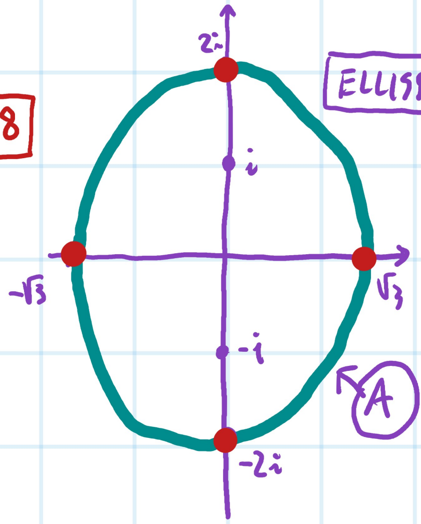
24 3, $-\frac{3+\sqrt{57}}{2}$, $-3i$, $4i$

25 $0, \ \pm(3+4i), \ \pm(4-3i), \ \pm\frac{7+i}{\sqrt{2}}, \ \pm\frac{1-7i}{\sqrt{2}}$

26 $\pm(1+i\sqrt{3}), \ \pm(\sqrt{3}-i), \ \pm\left(\frac{\sqrt{3}+1}{\sqrt{2}}+\frac{\sqrt{3}-1}{\sqrt{2}}i\right), \ \pm\left(\frac{\sqrt{3}-1}{\sqrt{2}}-\frac{\sqrt{3}+1}{\sqrt{2}}i\right)$

27 $1\pm 2i, \ \pm\sqrt[4]{\frac{3}{4}}, \ \pm i\sqrt[4]{\frac{3}{4}}, \ \frac{\sqrt{3}}{2}(1\pm i), \ \frac{\sqrt{3}}{2}(-1\pm i)$

28



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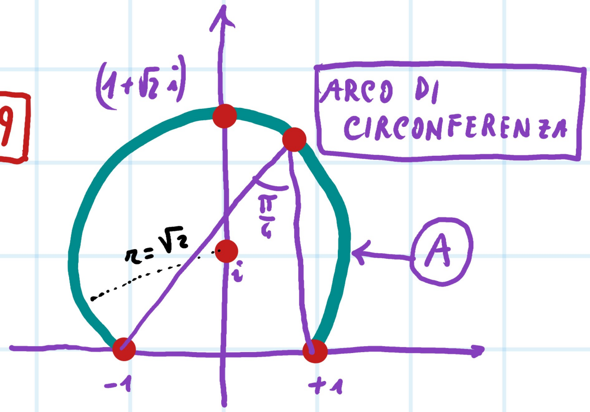

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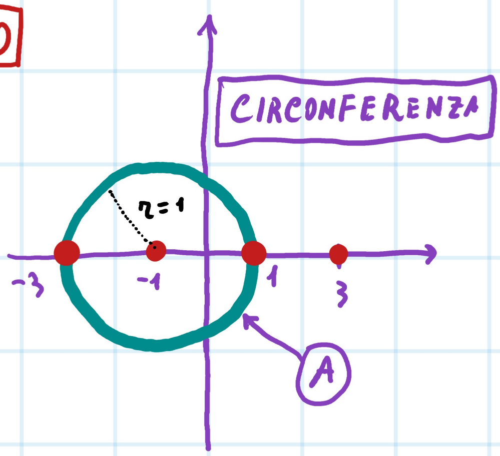

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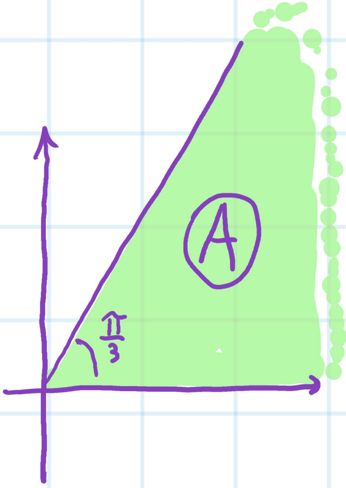

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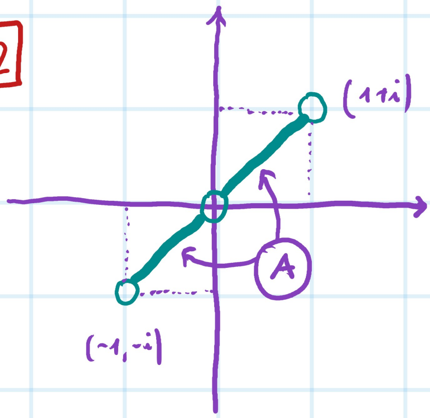

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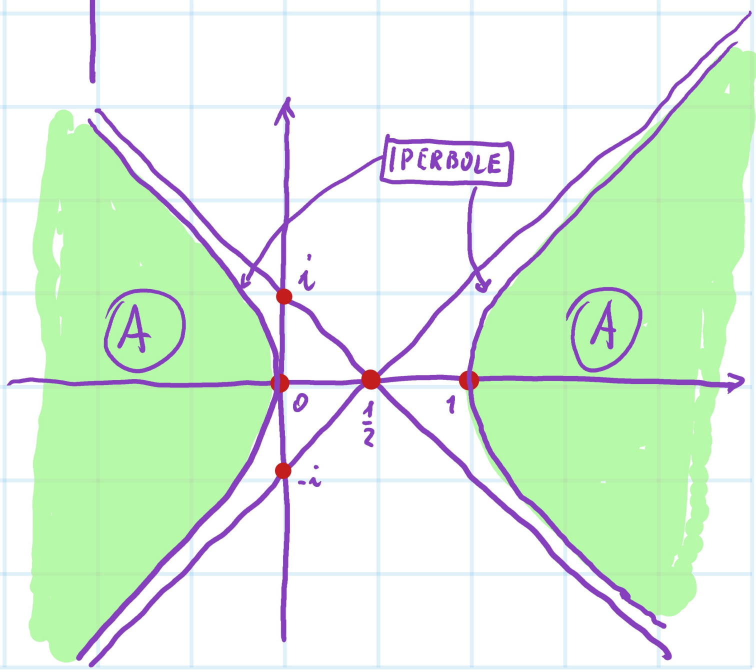

34 5

35 5

36 SOLO -1

37 8

38 $\partial A = \{ z \in \mathbb{C} \mid |z| = 1 \}$

39 CONVERGE SULL'INSIEME $\{ z \in \mathbb{C} \mid |z| \leq \frac{1}{27} \} - \{ \frac{1}{27} \}$

40 CONVERGE SULL'INSIEME $\{ z \in \mathbb{C} \mid |z| < e \}$

41 CONVERGE SULL'INSIEME $\{ z \in \mathbb{C} \mid |z| \leq 1 \}$

42 CONVERGE SULL'INSIEME $\{ z \in \mathbb{C} \mid |z| \leq 1 \} - \{ 1 \}$

Analisi Matematica 1º, a.a. 2020/21, 2° semestre
Corso di Laurea in Matematica
Docenti: Emanuele Callegari, Fabio Ciolli

Equazioni su $\mathbb{C}$

1. Trovare le soluzioni complesse delle equazioni seguenti, esprimendole in forma cartesiana, trigonometrica ed esponenziale.

1. $\left(z^2+i\right)^2=1\,.$
2. $z^2+\overline{z}+2z=1\,.$
3. $w^4-iw^3-w=-i\,.$
4. $z^6+\mathrm{i}z^3=0\,.$
5. $(z+\mathrm{i})^3=\dfrac{1-\mathrm{i}}{1+\mathrm{i}}\,.$
6. $(z+\sqrt{2})^2+\dfrac{4}{1+\mathrm{i}\sqrt{3}}=0\,.$
7. $\left(z^2\left|\dfrac{\overline{z}}{4}\right|+2\right)\left(z(i\,\overline{z}+2)-1\right)=0\,.$
8. $\left(z^2+\overline{z}+1\right)\left(z^2|z|+2\right)=0\,.$
9. $\left(z^3+|z|\right)\left(z^2+z+i\right)=0\,.$
10. $\left(z^3\,\overline{z}-|z|\right)\left(z^2+2i-1\right)=0\,.$

2. Dopo aver determinato il dominio delle seguenti equazioni complesse trovarne le soluzioni.

1. $\dfrac{\bar{z}}{2-z}+\dfrac{2|z|^2}{z^3-8}=0\,.$
2. $\dfrac{1}{|z|^2-1}\left(1-\dfrac{1}{z}\right)=\dfrac{1}{|z|}\,.$
3. $\left(z+|z|-1\right)\left(z^2+\dfrac{8}{z^2+4}\right)=0\,.$
4. $\left(z^3\,\overline{z}+\dfrac{i}{2}\right)\left(iz-\dfrac{2}{\overline{z}+1}\right)=0\,.$
5. $\left(2\overline{z}z^3-\dfrac{3}{2}+i\dfrac{\sqrt{27}}{2}\right)\left(2z-\dfrac{i}{\overline{z}}\right)=0\,.$
6. $\left(i\overline{z}-\dfrac{2}{z}\right)\left(z^2|z|+i(1+i)\right)=0\,.$

home Ciolli

Esercizi sui numeri complessi

Scrivere in forma algebrica z= a + ib con $a, b \in \mathbb{R}$ i seguenti numeri complessi:
1)

$$\frac{1}{i(3+2i)^2}$$

$$= \frac{1}{i(9+4i^2+12i)} = \frac{1}{i(9+12i-4)} = \frac{1}{9i+12i^2-4i} =$$

$$= -\frac{1}{12-5i} = \frac{1(12+5i)}{(12-5i)(12+5i)} = -\frac{-12-5i}{144+25} = -\frac{12}{169} - \frac{5}{169}i.$$

In questo esercizio, così come nei successivi, moltiplichiamo numeratore e denominatore per il coniugato del denominatore svolgendo poi alcuni passaggi algebrici. Ricordare che dato un numero complesso $z = a + ib$ il suo coniugato $\overline{z}$ è $a - ib$. Notare inoltre che $i^2 = -1$.

2)

$$\frac{(2+i)(1-i)}{3-2i}$$

$$= \frac{(2+i)(1-i)(3+2i)}{(3-2i)(3+2i)} = \frac{(2-2i+i-i^2)(3+2i)}{9-4i^2} =$$

$$= \frac{9+6i-6i-4i^2+3i+2i^2}{9+4} = \frac{11+3i}{13} = \frac{11}{13} + \frac{3}{13}i.$$

3)

$$\frac{(\sqrt{3}+\sqrt{2}i)^3}{\sqrt{2}-\sqrt{3}i}$$

$$= \frac{(\sqrt{3}+\sqrt{2}i)^3(\sqrt{2}+\sqrt{3}i)}{(\sqrt{2}-\sqrt{3}i)(\sqrt{2}+\sqrt{3}i)} = \frac{(3\sqrt{3}+9\sqrt{2}i-6\sqrt{3}-2\sqrt{2}i)(\sqrt{2}+\sqrt{3}i)}{2-3i^2} =$$

$$= \frac{(-3\sqrt{3}+7\sqrt{2}i)(\sqrt{2}+\sqrt{3}i)}{2+3} = \frac{-3\sqrt{6}-9i+14i-7\sqrt{6}}{5} = \frac{-10\sqrt{6}+5i}{5} = -2\sqrt{6}+i.$$

Ridurre in forma trigonometrica i seguenti numeri complessi:
1)

$$z = -3i$$

$$a = 0, b = -3 \rightarrow \varrho = \sqrt{0+9} = 3 \rightarrow \left.\begin{array}{l} \cos\theta = \frac{0}{3} = 0 \\ \\ \sin\theta = \frac{-3}{3} = -1 \end{array}\right\} \rightarrow \theta = \frac{3}{2}\pi$$

$$z = 3(\cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi).$$

Si ricorda che

$$z = a + ib: \qquad \varrho = \sqrt{a^2 + b^2}, \quad \cos\theta = \frac{a}{\varrho}, \quad sin\theta = \frac{b}{\varrho}, \quad z = \varrho(\cos\theta + i\sin\theta)$$

2)

$$z = -5$$

$$a = -5, b = 0 \to \varrho = \sqrt{25 + 0} = 5 \to \left.\begin{array}{l} \cos\theta = \frac{-5}{5} = -1 \\ \\ \sin\theta = \frac{0}{5} = 0 \end{array}\right\} \to \theta = \pi$$

$$z = 5(\cos\pi + i\sin\pi).$$

3)

$$z = \sqrt{3} + 1\mathrm{i}$$

$$a = \sqrt{3}, b = 1 \to \varrho = \sqrt{3 + 1} = 2 \to \left.\begin{array}{l} \cos\theta = \frac{\sqrt{3}}{2} \\ \\ \sin\theta = \frac{1}{2} \end{array}\right\} \to \theta = \frac{5}{6}\pi$$

$$z = 2(\cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi).$$

I seguenti numeri complessi non possono essere direttamente trasformati in forma trigonometrica. Moltiplichiamo quindi numeratore e denominatore per il coniugato del denominatore ed eseguiamo alcuni passaggi algebrici per ridurli alla forma $z = a + ib$.

1)

$$\frac{1+i}{1-i}$$

$$= \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{1 + i^2 + 2i}{2} = \frac{2i}{2} = i \text{ quindi } a = 0, b = 1$$

$$\varrho = \sqrt{0 + 1} = 1 \to \left.\begin{array}{l} \cos\theta = \frac{0}{1} = 0 \\ \\ \sin\theta = \frac{1}{1} = 1 \end{array}\right\} \to \theta = \frac{\pi}{2}$$

$$z = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}.$$

2)

$$\frac{1+i}{\sqrt{3}+i}$$

$$= \frac{(1+i)(\sqrt{3}-i)}{(\sqrt{3}+i)(\sqrt{3}-i)} = \frac{\sqrt{3}+1}{4} + \frac{i(\sqrt{3}-1)}{4}$$

$$\text{quindi } a = \frac{\sqrt{3}+1}{4}, b = \frac{(\sqrt{3}-1)}{4}$$

$$\varrho = \sqrt{\frac{3+1+2\sqrt{3}}{16} + \frac{3+1-2\sqrt{3}}{16}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\left.\begin{array}{l} \cos\theta = \frac{\sqrt{3}+1}{4} \cdot \sqrt{2} = \frac{\sqrt{6}+\sqrt{2}}{4} \\ \sin\theta = \frac{\sqrt{3}-1}{4} \cdot \sqrt{2} = \frac{\sqrt{6}-\sqrt{2}}{4} \end{array}\right\} \to \theta = \frac{\pi}{12}$$

$$\text{Notare che } \cos\frac{\pi}{12} = \cos(\frac{\pi}{4} - \frac{\pi}{6}) = \cos\frac{\pi}{4} \cdot \sin\frac{\pi}{6} + \sin\frac{\pi}{4} \cdot \cos\frac{\pi}{6} = \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$$

$$z = \frac{1}{\sqrt{2}}(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}).$$

3)

$$\frac{1+i\sqrt{3}}{1-i}$$

$$= \frac{(1+i\sqrt{3})(1+i)}{(1-i)(1+i)} = \frac{(1+i\sqrt{3})(1+i)}{1-i^2} = \frac{1+i+\sqrt{3}i+\sqrt{3}i^2}{2} = \frac{1-\sqrt{3}}{2} + \frac{i(1+\sqrt{3})}{2}$$

$$\text{Quindi } a = \frac{1-\sqrt{3}}{2}, b = \frac{1+\sqrt{3}}{2}$$

$$\varrho = \sqrt{\frac{1+3-2\sqrt{3}}{4} + \frac{3+1+2\sqrt{3}}{4}} = \sqrt{2}$$

$$\left.\begin{array}{l} \cos\theta = \frac{1-\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}} \\ \sin\theta = \frac{1+\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} = \frac{1+\sqrt{3}}{2\sqrt{2}} \end{array}\right\} \to \theta = \frac{7}{12}\pi$$

$$z = \sqrt{2}(\cos\frac{7}{12}\pi + i\sin\frac{7}{12}\pi)$$

Utilizzare la formula di De Moivre ($z^n = \varrho^n(\cos(n\theta) + i\sin(n\theta))$ per calcolare le potenze dei seguenti numeri complessi :

1)

$$(\frac{1-i}{1+i})^3$$

$$= (\frac{(1-i)^2}{1-i^2})^3 = (\frac{1+i^2-2i}{2})^3 = (\frac{-2i}{2})^3 = -i^3$$

Quindi $a = 0, b = -1$

$$\varrho = \sqrt{0+1} = 1 \qquad \left.\begin{array}{l} \cos\theta = \frac{0}{1} = 0 \\ \\ \sin\theta = \frac{-1}{1} = -1 \end{array}\right\} \rightarrow \theta = \frac{3}{2}\pi$$

$$(\frac{1-i}{1+i})^3 = 1^3(\cos(3\cdot\frac{3}{2}\pi) + \sin(3\cdot\frac{3}{2}\pi)i) = \cos\frac{9}{2}\pi + \sin\frac{9}{2}\pi i = 0 + 1i = i$$

2)

$$(1+i)^{20}$$

$$a = 1, \quad b = 1 \qquad \varrho = \sqrt{2}$$

$$\left.\begin{array}{l} \cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ \\ \sin\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \end{array}\right\} \rightarrow \theta = \frac{3}{2}\pi$$

$$(1+i)^{20} = (\sqrt{2})^{20}(\cos(20\cdot\frac{\pi}{4}\pi) + \sin(20\cdot\frac{\pi}{4})i) = 2^{10}(-1+0i) = -2^{10}$$

3)

$$(1-i)^{11}$$

$$a = 1, \quad b = -1 \qquad \varrho = \sqrt{2}$$

$$\left.\begin{array}{l} \cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ \\ \sin\theta = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \end{array}\right\} \rightarrow \theta = \frac{\pi}{4}$$

$$(1-i)^{11} = (\sqrt{2})^{11}(\cos(11\cdot\frac{7}{4}\pi)+\sin(11\cdot\frac{7}{4}\pi)i) = 2^5\sqrt{2}(\cos\frac{5}{4}\pi-\sin 54\pi i) = 32\sqrt{2}(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}) = -32-32i$$

Utilizzare la formula $z_k = \varrho^{\frac{1}{n}}[\cos(\frac{\theta+2k\pi}{n}) + i\sin(\frac{\theta+2k\pi}{n})]$ per trovare le radici dei seguenti numeri complessi:

1)

$$\sqrt[3]{i-1}$$

$$a=-1, \quad b=1, \qquad \varrho=\sqrt{2} \qquad \left.\begin{array}{l} \cos\theta = \frac{-1}{\sqrt{2}} \\ \\ \sin\theta = \frac{1}{\sqrt{2}} \end{array}\right\} \rightarrow \theta = \pi - \frac{\pi}{4} = \frac{3}{4}\pi$$

$$z_k = \sqrt{2}^{\frac{1}{3}}[\cos(\frac{\frac{3}{4}\pi+2k\pi}{3}) + i\sin(\frac{\frac{3}{4}\pi+2k\pi}{3})] \text{ per k = 0, 1, 2}$$

$$z_0 = \sqrt{2}^{\frac{1}{3}}[\cos(\frac{\frac{3}{4}\pi}{3}) + i\sin(\frac{\frac{3}{4}\pi}{3})] = \sqrt[6]{2}[\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}]$$

$$z_1 = \sqrt[6]{2}[\cos(\frac{\frac{3}{4}\pi+2\pi}{3}) + i\sin(\frac{\frac{3}{4}\pi+2\pi}{3})] = \sqrt[6]{2}[\cos(\frac{\frac{11}{4}\pi}{3}) + i\sin(\frac{\frac{11}{4}\pi+2\pi}{3})] =$$

$$= \sqrt[6]{2}[cos(\frac{11}{12}\pi) + i\sin(\frac{11}{12}\pi)] = \sqrt[6]{2}[-\frac{\sqrt{6}+\sqrt{2}}{4} + i\frac{\sqrt{6}-\sqrt{2}}{4}]$$

$$\text{Essendo } \cos\frac{11}{12}\pi = \cos(\pi - \frac{\pi}{12}) = -\cos\frac{\pi}{12} = -\frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\text{e } \sin\frac{11}{12}\pi = \sin(\pi - \frac{\pi}{12}) = \sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$z_2 = \sqrt[6]{2}[\cos(\frac{\frac{3}{4}\pi+4\pi}{3}) + i\sin(\frac{\frac{3}{4}\pi+4\pi}{3})] = \sqrt[6]{2}[cos(\frac{19}{12}\pi) + i\sin(\frac{19}{12}\pi)] =$$

$$= \sqrt[6]{2}[\frac{\sqrt{6}-\sqrt{2}}{4} + -i\frac{\sqrt{6}+\sqrt{2}}{4}]$$

$$\text{Essendo } \frac{19}{12}\pi = \frac{3}{2}\pi + \frac{\pi}{12} \text{ abbiamo } \cos(\frac{3}{2}\pi + \frac{\pi}{12}) = \sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\text{e } \sin(\frac{3}{2}\pi + \frac{\pi}{12}) = -\cos\frac{\pi}{12} = -\frac{\sqrt{6}+\sqrt{2}}{4} \text{ da cui la soluzione.}$$

2)

$$\sqrt[4]{-2-2\sqrt{3}i}$$

$$a = -2, \qquad b = -2\sqrt{3}, \qquad \varrho = \sqrt{4+12} = 4 \qquad \left.\begin{array}{l} \cos\theta = -\frac{1}{2} \\ \\ \sin\theta = -\frac{\sqrt{3}}{2} \end{array}\right\} \to \theta = \frac{4}{3}\pi$$

$$z_k = 4^{\frac{1}{4}}[\cos(\frac{\frac{4}{3}\pi + 2k\pi}{4}) + i\sin(\frac{\frac{4}{3}\pi + 2k\pi}{4})] \text{ per k = 0, 1, 2, 3}$$

$$z_0 = 4^{\frac{1}{4}}[\cos(\frac{1}{3}\pi + i\sin\frac{1}{3}\pi] = 4^{\frac{1}{4}}[\frac{1}{2} + i\frac{\sqrt{3}}{2}]$$

$$z_1 = 4^{\frac{1}{4}}[\cos(\frac{5}{6}\pi + i\sin\frac{5}{6}\pi] = 4^{\frac{1}{4}}[\cos(\frac{\pi}{6} - i\sin\frac{\pi}{6}] = 4^{\frac{1}{4}}[-\frac{\sqrt{3}}{2} + \frac{i}{2}]$$

$$z_2 = 4^{\frac{1}{4}}[\cos(\frac{4}{3}\pi + i\sin\frac{4}{3}\pi] = 4^{\frac{1}{4}}[-\cos(\frac{\pi}{3} + i(-\sin\frac{\pi}{3})] = 4^{\frac{1}{4}}[-\frac{1}{2} - i\frac{\sqrt{3}}{2}]$$

$$z_3 = 4^{\frac{1}{4}}[\cos(\frac{11}{6}\pi + i\sin\frac{11}{6}\pi] = 4^{\frac{1}{4}}[-\cos(\frac{\pi}{6} + i(-\sin\frac{\pi}{6})] = 4^{\frac{1}{4}}[\frac{\sqrt{3}}{2} - \frac{i}{2}]$$

Risolvere le seguenti equazioni con i numeri complessi:

1)
$$z^2 + 2z + 3 = 0$$

$$z = \frac{-1 \pm \sqrt{1-3}}{1}$$

$$z_1 = -1 - i\sqrt{2}; \; z_2 = -1 + i\sqrt{2}$$

Notare che $\sqrt{-2} = \sqrt{-1 \cdot 2} = \sqrt{-1} \cdot \sqrt{2} = \sqrt{i^2} \cdot \sqrt{2} = i\sqrt{2}$

2)
$$z + 3i + Re(z)(i + (Im(z))^2) = 0$$

Sostituendo $z = a + ib$ abbiamo

$a + ib + 3i + a(i + b^2) = 0 \rightarrow a + ib + 3i + ai + ab^2 = 0 \rightarrow a + ib^2 + i(b + 3 + a) = 0$

$$\begin{cases} a + ab^2 = 0 \\ b + 3 + a = 0 \end{cases} \rightarrow \begin{cases} a = 0 \quad \cup \quad 1 + b^2 = 0 \\ b + a = -3 \end{cases} \rightarrow \begin{matrix} \begin{cases} a = 0 \\ b = -3 \end{cases} \rightarrow z = -3i \\ \begin{cases} b = \sqrt{-1} \\ a + b = -3 \end{cases} \rightarrow \text{nessuna soluzione reale} \end{matrix}$$

3)
$$z^2 + 2iz - 3 = 0$$

dato che il coefficiente di è b pari, usiamo la formula ridotta $z = \dfrac{-\frac{b}{2} \pm \sqrt{(\frac{b}{2})^2 - ac}}{a}$

$$z = -1 \pm \sqrt{i^2 + 3} = -i \pm \sqrt{1 + 3} \rightarrow \begin{matrix} z_1 = \sqrt{2} - i \\ z_2 = -\sqrt{2} - i \end{matrix}$$

4)
$$iz^3 = \overline{z}$$

Dato $z = \varrho(\cos\theta + i\sin\theta)$ abbiamo $z^3 = \varrho^3(\cos 3\theta + i \sin 3\theta)$ e $\overline{z} = \varrho(\cos\theta - i\sin\theta) \rightarrow$

$$i\varrho^3(\cos 3\theta + i\sin 3\theta) = \varrho(\cos\theta - i\sin\theta) \rightarrow$$
$$\varrho^3(i\cos 3\theta + i^2\sin 3\theta) = \varrho(\cos\theta - i\sin\theta) \rightarrow$$
$$\varrho^3(-\sin 3\theta + i\cos 3\theta) = \varrho(\cos\theta - i\sin\theta)$$

Definendo le condizioni per l'uguaglianza di due numeri complessi in forma trigonometrica abbiamo

$$\begin{cases} \varrho^3 = \varrho \\ \cos 3\theta = -\sin\theta \\ -\sin 3\theta = \cos\theta \end{cases} \rightarrow \begin{cases} \varrho^3 - \varrho = 0 \\ -\cot 3\theta = -\text{tg}\theta \end{cases} \rightarrow \begin{cases} \varrho = 0 \quad \cup \quad \varrho = \left\langle \begin{array}{l} +1 \\ -1 \text{ non ammissibile} \end{array}\right. \\ \text{tg}(\frac{\pi}{2} - 3\theta) = \text{tg}\theta \end{cases} \rightarrow$$

$$\begin{cases} \varrho = 0 \quad \cup \quad \varrho = 1 \\ \text{tg}(\frac{\pi}{2} - 3)\theta = \theta + k\pi \qquad \text{notare che } \text{tg}x = \text{tg}\beta \Leftrightarrow x = \beta + k\pi \end{cases} \rightarrow$$

$$\begin{cases} \varrho = 0 \vee \varrho = 1 \\ \frac{\pi}{2} - 4\theta = k\pi \leftrightarrow 4\theta = \frac{\pi}{2} - k\pi \leftrightarrow \theta = \frac{\pi - 2k\pi}{8} \qquad \text{per k = 0,1,2} \end{cases}$$

$$z_1 = 0$$
$$z_2 = \cos(\tfrac{\pi-2k\pi}{8}) + i\sin(\tfrac{\pi-2k\pi}{8}) \text{ per k = 0,1,2}$$

5)

$$z^6 + 2z^3 - 3 = 0$$

Ponendo $w = z^3$ abbiamo $w^2 + 2w - 3 = 0 z^3 = -1 \pm \sqrt{1+3} = \left\langle \begin{array}{l} z^3 = -3 \\ z^3 = 1 \end{array}\right.$

Per $z^3 = -3$ $\varrho = 3$

$$\left.\begin{array}{r} \cos\theta = \frac{-3}{3} \\ \sin\theta = 0 \end{array}\right\} \theta = \pi$$

$$z_1 = \sqrt[3]{3}(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$$
$$z_2 = \sqrt[3]{3}(\cos\pi + i\sin\pi)$$
$$z_3 = \sqrt[3]{3}(\cos\frac{5}{3}\pi + i\sin\frac{5}{3\pi})$$

Per $z^3 = 1$ $\varrho = 1$

$$\left.\begin{array}{r} \cos\theta = 1 \\ \sin\theta = 0 \end{array}\right\} \theta = 0$$

$$z_4 = \sqrt[3]{1}(\cos 0 + i\sin 0)$$
$$z_5 = \sqrt[3]{1}(\cos\frac{2}{3}\pi + i\sin 23\pi)$$
$$z_6 = \sqrt[3]{1}(\cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi)$$

6)

$$z^2 + \overline{z} = 0$$

Sostituendo $z = a + ib$ abbiamo

$$(a+ib)^2 + a - ib = 0 \to a^2 - b^2 + 2aib + a - ib = 0$$

$$\begin{cases} a^2 - b^2 + a = 0 \\ 2ab - b = 0 \end{cases} \to \begin{cases} a^2 - b^2 + a = 0 \\ b(2a-1) = 0 \end{cases} \to$$

$$\begin{cases} a^2 - b^2 + a = 0 \\ b = 0 \quad \cup \quad a = \frac{1}{2} \end{cases} \to \begin{matrix} \begin{cases} a^2 + a = 0 \\ b = 0 \end{cases} \to \begin{cases} a(a+1) = 0 \left\langle \begin{matrix} a = -1 \\ a = 0 \end{matrix} \right. \\ b = 0 \end{cases} \to \begin{matrix} z_1 = -1 \\ z_2 = 0 \end{matrix} \\ \begin{cases} (\frac{1}{2})^2 - b^2 + \frac{1}{2} = 0 \\ a = \frac{1}{2} \end{cases} \to \begin{cases} b^2 = \frac{3}{4} \left\langle \begin{matrix} b = -\frac{\sqrt{3}}{2} \\ b = \frac{\sqrt{3}}{2} \end{matrix} \right. \\ a = \frac{1}{2} \end{cases} \to \begin{matrix} z_3 = \frac{1}{2} - i\frac{\sqrt{3}}{2} \\ z_4 = \frac{1}{2} + i\frac{\sqrt{3}}{2} \end{matrix} \end{matrix}$$

7)

$$\overline{z}^4 = |z|$$

$$\varrho^4(\cos 4\theta - i \sin 4\theta) = \varrho$$

$$\begin{cases} \varrho^4 = \varrho \\ 4\theta = 2k\pi \end{cases} \to \begin{cases} \varrho(\varrho^3 - 1) = 0 \\ \theta = \frac{k\pi}{2} \end{cases} \to \begin{cases} \varrho = 0 \quad \cup \quad \varrho = 1 \\ \theta = \frac{k\pi}{2} \end{cases}$$

$$\begin{aligned} & z_0 = 0 \\ & z_1 = \cos \tfrac{k\pi}{2} + i \sin \tfrac{k\pi}{2} \text{ per } k = 0,1,2,3 \end{aligned}$$

8)

$$iRe(z) + z^2 = |z|^2 + 1$$

Sostituendo $|z| = \sqrt{a^2 + b^2}$ e $z = a + ib$ abbiamo

$$ai + a^2 - b^2 + 2aib - a^2 - b^2 - 1 = 0 \quad \to \quad a^2 - b^2 - a^2 - b^2 - 1 + i(a + 2ab) = 0$$

$$\begin{cases} a^2 - b^2 - a^2 - b^2 - 1 = 0 \\ a + 2b = 0 \end{cases} \to \begin{cases} -2b^2 - 1 = 0 \\ a(1+2b) = 0 \end{cases} \to \begin{cases} 2b^2 = -1 \\ a = 0 \quad \cup \quad b = -\frac{1}{2} \end{cases} \to$$

$$\begin{cases} b = \sqrt{-\frac{1}{2}} \\ a = 0 \quad \cup \quad b = -\frac{1}{2} \end{cases} \to \varnothing$$

9)

$$2|z|^2 = z^3$$

$$2\varrho^2 = \varrho^3(\cos 3\theta + i \sin 3\theta)$$

$$\begin{cases} 2\varrho^2 = \varrho^3 \\ 2k\pi = 3\theta \end{cases} \to \begin{cases} 2\varrho^2 - \varrho^3 = 0 \\ 2k\pi = 3\theta \end{cases} \to \begin{cases} \varrho^2(2 - \varrho) = 0 \left\langle \begin{matrix} \varrho = 0 \\ \varrho = 2 \end{matrix} \right. \\ \theta = \frac{2k\pi}{3} \text{ per } k = 0,1,2 \end{cases}$$

$$\begin{aligned} & z_1 = 0 \\ & z_2 = 2(\cos \tfrac{2k\pi}{3} + i \sin \tfrac{2k\pi}{3}) \quad \text{per } k = 0,1,2 \end{aligned}$$

10)

$$z^2 + im(z) + 2\overline{z} = 0 \text{ dove z = a + ib}$$

$$\Leftrightarrow (a+ib)^2 + ib + 2(a-ib) = 0$$

$$\Leftrightarrow a^2 - b^2 + 2iab + ib - 2a - 2ib = 0$$

$$\Leftrightarrow a^2 - b^2 + 2a + i(2ab - b) = 0$$

$$\Leftrightarrow \begin{cases} a^2 - b^2 + 2a = 0 \\ 2ab - b = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} a^2 - b^2 + 2a = 0 \\ b(2a-1) = 0 \end{cases} \Leftrightarrow a = \frac{1}{2} \quad \cup \quad b = 0$$

Se $a = \frac{1}{2} \to \frac{1}{4} - b^2 + 1 = 0 \Rightarrow b = \pm\frac{\sqrt{5}}{2} \Rightarrow \qquad z_1 = \frac{1}{2} - i\frac{\sqrt{5}}{2}, \quad z_2 = \frac{1}{2} + i\frac{\sqrt{5}}{2}$

Se $b = 0 \to \quad a^2 + 2b = 0 \Rightarrow \quad a = 0 \quad a = -2 \Rightarrow \qquad z_3 = 0, \quad z_4 = -2$