ESERCITAZIONI DI ANALISI 1

Exe. 5

21 OTTOBRE 2022

ESERCIZI PROPOSTI

Quali, tra le affermazioni che seguono, equivalgono a dire che $\lim_{n\to+\infty} a_n = \ell \in \mathbf{R}$?

1) per ogni $\epsilon > 0$, definitivamente in n, si ha $|a_n - \ell| \leq \epsilon$ SI
2) per ogni $\epsilon \geq 0$, definitivamente in n, si ha $|a_n - \ell| < \epsilon$ NO
3) per ogni $\epsilon > 0$, definitivamente in n, si ha $|a_n - \ell| < \frac{\epsilon}{2}$ SI
4) per ogni $\epsilon \in (0, 1]$, definitivamente in n, si ha $|a_n - \ell| < \epsilon$ SI
5) COME (1) MA $\boxed{ = \text{DEF. IN } k}$ E $\boxed{ = |a_n - \ell| < \frac{1}{k}}$ SI
6) COME (1) MA $\boxed{ = |\sin(a_n - \ell)| < \varepsilon}$. NO

DIRE SE LE SEGUENTI AFFERMAZIONI SONO VERE O FALSE.

SE VERE DIMOSTRARLE, SE FALSE ESIBIRE UN CONTROESEMPIO.

7) $(a_n \to 0) \Leftrightarrow (|a_n| \to 0)$ VERA

8) $(a_n \to \ell) \Leftrightarrow (|a_n| \to |\ell|)$ FALSA

9) $\Big((a_n \to +\infty) \text{ E } ((b_n) \text{ LIMITATA}) \Big) \Rightarrow (a_n + b_n \to +\infty)$ **VERA**

10) $\Big((a_n \to \ell) \text{ E } (\forall n \in \mathbb{N} \; a_n > 0) \Big) \Rightarrow (\ell > 0)$ **FALSA**

11) COME **(10)** MA $\boxed{>}$ È $\boxed{\geq}$. **VERA**

12) $\Big((C \subset \mathbb{R}) \text{ E } (a_n \to \ell) \text{ E } (\forall n \in \mathbb{N} \; a_n \in C) \Big) \Rightarrow (\ell \in C)$ **FALSA**

13) COME **(12)** MA AGGIUNGENDO L'IPOTESI CHE C È CHIUSO **VERA**

14) $\Big((C \subset \mathbb{R}) \text{ E } (C \text{ NON CHIUSO}) \Big) \Rightarrow \big(\exists (a_n) \text{ A VALORI IN } C \text{ T.C. } a_n \to \ell \notin C \big)$ **VERA**

15) $\Big((a_n \to 0) \text{ E } (a_n \text{ DECRESCE}) \Big) \Rightarrow (\forall n \; a_n \geq 0)$ **VERA**

16) $\Big((a_n \to 0) \text{ E } (\forall n \in \mathbb{N} \; a_n \geq 0) \Big) \Rightarrow (a_n \text{ DECRESCE})$ **FALSA**

USANDO SOLO LA DEFINIZIONE (ED EVENTUALMENTE QUALCHE TRUCCO) MOSTRARE CHE:

17) $\displaystyle \lim_{n \to +\infty} \frac{n^7 - n^3}{10 + n^4} = +\infty$

18) $\displaystyle \lim_{n \to +\infty} \frac{1}{n + \sin n} = 0$

19) $\displaystyle \lim_{n \to +\infty} \frac{n^7 - 3n^2}{n^7 + 8n} = 1$

20) $\displaystyle \lim_{n \to +\infty} \sqrt[n]{3 + \sin n} = 1$

SOLUZIONI

Quali, tra le affermazioni che seguono, equivalgono a dire che $\lim_{n\to+\infty} a_n = \ell \in \mathbf{R}$?

B 7 per ogni $\epsilon > 0$, definitivamente in n, si ha $|a_n - \ell| \leq \epsilon$ SI

8 per ogni $\epsilon \geq 0$, definitivamente in n, si ha $|a_n - \ell| < \epsilon$ NO

9 per ogni $\epsilon > 0$, definitivamente in n, si ha $|a_n - \ell| < \frac{\epsilon}{2}$ SI

10 per ogni $\epsilon \in (0,1]$, definitivamente in n, si ha $|a_n - \ell| < \epsilon$

A $\forall \varepsilon > 0$ DEF. IN n $|a_n - \ell| < \varepsilon$

A $\Rightarrow$ B

$$\frac{\varepsilon}{2} < \varepsilon$$

$$|a_n - \ell| \leq \frac{\varepsilon}{2} < \varepsilon$$

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19) $\lim_{n\to+\infty} \frac{n^7 - 3n^2}{n^7 + 8n} = 1$

$\forall \varepsilon > 0$ DEF. IN n $\left| \frac{n^7 - 3n^2}{n^7 + 8n} - 1 \right| < \varepsilon$

$$\begin{cases} \cancel{\dfrac{n^7-3n^2}{n^7+8n} < 1+\varepsilon} \\[2ex] \dfrac{n^7-3n^2}{n^7+8n} > 1-\varepsilon \end{cases}$$

$$\frac{\cancel{n^7+8n}-8n-3n^2}{n^7+8n} > \cancel{1}-\varepsilon$$

$$\frac{-8-3n}{n^6+8} > -\varepsilon$$

$$\boxed{\frac{3n+8}{n^6+8} < \boxed{\varepsilon}}$$

$n \geq 9$

$$\frac{3n+8}{n^6+8} < \frac{4n}{n^6+8} < \frac{4n}{n^6} = \frac{4}{n^5} < \frac{1}{n^4} < \boxed{\frac{1}{n}}$$

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17) $\lim\limits_{n\to+\infty} \dfrac{n^7-n^3}{10+n^4} = +\infty$

$\forall M > 0 \quad$ DEF. IN $n \quad \boxed{\frac{n^7 - n^3}{10 + n^4} > M}$

$n^3(n-1) > 1$

$n^4 - n^3 > 1$

DEF. IN n

$$\boxed{\frac{n^7 - n^3}{n^4 + 10}} = \frac{n^3(n^4-1)}{n^4+10} > \frac{n^3 \cdot n^3}{n^4+10} = \frac{n^6}{n^4+10} > \frac{n^6}{n^4+n^4} =$$

$$= \frac{n^6}{2n^4} > \frac{n^5}{n^4} = \boxed{n} > M$$

$$= \frac{n^2}{2} \geq n$$

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18) $\lim_{n \to +\infty} \frac{1}{n + \sin n} = 0$

$\forall \varepsilon > 0 \quad$ DEF. IN $n \quad \left| \frac{1}{n + \sin n} \right| < \varepsilon$

$$\begin{cases} \boxed{\frac{1}{n + \sin n} < \varepsilon} \\ \cancel{\frac{1}{n + \sin n} > -\varepsilon} \end{cases}$$

SI

$n + \sin n > \frac{1}{\varepsilon}$

$n + \sin n > n - 1$

?

$\underbrace{n - 1 > \frac{1}{\varepsilon}}$ DEF. INN

$n > 1 + \frac{1}{\varepsilon}$ DEF INN

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20) $\lim_{n \to +\infty} \sqrt[n]{3 + \sin n} = 1$

$\forall \varepsilon > 0$ DEF. INN $\left| \sqrt[n]{3 + \sin n} - 1 \right| < \varepsilon$

$$\begin{cases} \sqrt[n]{3 + \sin n} < 1 + \varepsilon \\ \boxed{\sqrt[n]{3 + \sin n} > 1 - \varepsilon} \end{cases}$$

GRATIS

$\boxed{3 + \sin n < (1 + \varepsilon)^n}$

$\boxed{4 < (1 + \varepsilon)^n}$

$$\boxed{(1+\varepsilon)^n > 4}$$

$$\boxed{(1+\varepsilon)^n} > \boxed{1+n\varepsilon} > 4$$

SEMPRE

DEF.

$$\boxed{1+n\varepsilon > 4}$$

$$\boxed{n > \boxed{\frac{3}{\varepsilon}}}$$

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7) $(a_n \to 0) \overset{(?)}{\Leftrightarrow} (|a_n| \to 0)$

$\to \forall \varepsilon > 0$ DEF. IN n $|a_n| < \varepsilon$

$\to \forall \varepsilon > 0$ DEF. IN n $\underbrace{\big||a_n| - 0\big|}_{|a_n|} < \varepsilon$

$$\underbrace{\Big(\underbrace{(a_n \to +\infty)}_{①} \text{ E } \underbrace{\big((b_n) \text{ INFERIORMENTE LIMITATA}\big)}\Big)} \Rightarrow \underbrace{(a_n + b_n \to +\infty)}_{②}$$

1. $\exists \lambda \in \mathbb{R}$ t.c. $\forall n \in \mathbb{N}$ $b_n \geq \lambda$

2. $\forall M > 0$ DEF. IN n $\boxed{a_n \geq M}$

3. $\forall M > 0$ DEF. IN n $\boxed{a_n + b_n > M}$

??

$$a_n > M - b_n > \overbrace{M - \lambda}^{M'}$$

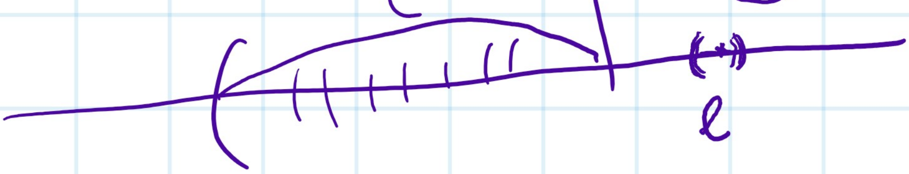
$a_n = n \to +\infty$ $\quad a_n + b_n = n - n^2 = n(1-n) < -n \to -\infty$

$b_n = -n^2 \to -\infty$

$a_n = \overline{n^2 + (-1)^n} \to +\infty$ $\quad a_n + b_n = (-1)^n$ NON HA LIMITE

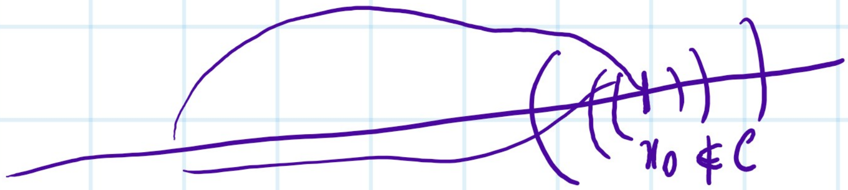
$b_n = -n^2 \to -\infty$

$$\Big((C \text{ È CHIUSO}) \wedge (a_n \in C \ \forall n) \wedge (a_n \to \ell) \Big) \Rightarrow (\ell \in C)$$



$a_n \in I \cap C^c$ DEF. INN

$a_n \notin C$ DEF INN



$\forall I$ int. di x_0

$I \cap C \neq \emptyset$

$|a_n - x_0| < \frac{1}{n}$

$I_n = (x_0 - \frac{1}{n}, x_0 + \frac{1}{n}) \qquad \forall n \ I_n \cap C \neq \emptyset$

$\forall n \in \mathbb{N}$ PRENDO $a_n \in \boxed{I_n \cap C}$

$\forall n \ a_n \in C$

PER CASA: DATE (a_n) E (b_n) DEFINITE DA:

$$\begin{cases} a_0 = 150 \\ b_0 = 294 \\ a_{n+1} = \dfrac{2}{\frac{1}{a_n} + \frac{1}{b_n}} \\ b_{n+1} = \dfrac{a_n + b_n}{2} \end{cases}$$

TROVARE (SE ESISTONO) $\lim_{n \to +\infty} a_n$ E $\lim_{n \to +\infty} b_n$