Esercitazioni di Analisi 1

Exe 7

4 Novembre 2022

PROBLEMI PROPOSTI (DA CAP. 2 ESERCIZIARIO)

(TUTTI I PROBLEMI VANNO RISOLTI SENZA USARE LA FORMULA DI STIRLING)

44 $\lim_{n\to+\infty} \frac{n^5 2^n + n^{10}}{n^2 + 3^n}$

51 $\lim_{n\to+\infty} \frac{\sqrt{n^{n+1}} + 2^{n^2}}{(2n)!}$

94 $\lim_{n\to+\infty} \sqrt[n]{n}$

Teorema 24 [Algebra degli o-piccoli] Siano date le successioni (a_n), (b_n), (A_n) e (B_n) tutte infinitesime (o infinite) per $n \to +\infty$. Allora valgono le regole:

(102) se $a_n = o(A_n)$ e $A_n = o(B_n)$ allora $a_n = o(B_n)$;

(103) se $a_n = o(A_n)$ allora $A_n + a_n \approx A_n$;

(104) se (A_n) e (B_n) hanno lo stesso ordine di infinito (o di infinitesimo) allora è equivalente affermare che $a_n = o(A_n)$ e che $a_n = o(B_n)$;

(105) se $a_n = o(A_n)$ e $b_n = o(A_n)$ allora $\alpha a_n + \beta b_n = o(A_n)$ per ogni $\alpha, \beta \in \mathbf{R}$;

(106) se $a_n = o(A_n)$ allora $a_n b_n = o(A_n b_n)$;

(107) se $a_n = o(A_n)$ e $b_n = o(B_n)$ allora $a_n b_n = o(A_n B_n)$;

PER ORA SOLO QUESTI

(108) se $a_n \approx A_n$ e $b_n \approx B_n$ allora $\lim_{n\to+\infty} \frac{a_n}{b_n} = \lim_{n\to+\infty} \frac{A_n}{B_n}$;

(109) se $a_n = o(A_n)$ e $b_n = o(B_n)$ allora $\lim_{n\to+\infty} \frac{A_n + a_n}{B_n + b_n} = \lim_{n\to+\infty} \frac{A_n}{B_n}$.

74 $\lim_{n\to+\infty} \left(1 + \frac{1}{n+2}\right)^n$

69 $\lim_{n\to+\infty} \left(1 - \frac{3}{n}\right)^n$

81 $\lim_{n\to+\infty} \left(\frac{n+9}{n+5}\right)^{3n}$

Teorema 26 Data una successione (a_n), se $a_n \to \pm\infty$ allora $\left(1 + \frac{1}{a_n}\right)^{a_n} \to e$ mentre $\left(1 - \frac{1}{a_n}\right)^{a_n} \to \frac{1}{e}$.

86 $\lim_{n\to+\infty} \left(\frac{n^2 - n - 6}{n^2 - n + 2}\right)^{\frac{n^4+n+3}{n^2-1}}$

84 $\lim_{n\to+\infty} \left(\frac{n^2 + n - 6}{n^2 + n + 7}\right)^n$

108 $\lim_{n\to+\infty} \frac{(n+1)^{n+\frac{1}{n}}+n!}{(n+2)^n-n^n}$

113 $\lim_{n\to+\infty} \frac{(n!)^{n+1}+((n+1)!)^n}{((n+1)!)^{n+\frac{1}{n}}+((n-1)!)^{n+2}}$

114 $\lim_{n\to+\infty} \left((n+9)^{100}+n^{98}\ln(n!)-n^{100}\right)\cdot\left(\sqrt{n^{199}+1}-\sqrt{n^{199}-1}\right)$

123 $\lim_{n\to+\infty} \frac{2^{n!}}{(2^n)!}$

124 $\lim_{n\to+\infty} \frac{(n!)!}{n^{n^n}}$

119 $\lim_{n\to+\infty} \frac{(n!)^2}{n^n}$

Teorema 36 [Rapporto] Sia (a_n) una successione a termini strettamente positivi e sia $\lambda \in \mathbf{R}$. Allora:

(130) se $\lambda > 1$ e definitivamente in n si ha $\frac{a_{n+1}}{a_n} > \lambda$, allora $a_n \to +\infty$;

(131) se $0 < \lambda < 1$ e definitivamente in n si ha $\frac{a_{n+1}}{a_n} < \lambda$ allora $a_n \to 0$.

117 $\lim_{n\to+\infty} \frac{(n!)^2\cdot 5^n}{(2n)!}$

118 $\lim_{n\to+\infty} \frac{(n!)^2\cdot 4^n}{(2n)!}$

PROBLEMA "CREATIVO" DA RISOLVERE PRIMA DI VENERDÌ 11 NOVEMBRE

TROVARE TUTTI I PUNTI LIMITE (IN PARTICOLARE liminf E limsup) DELLA SEGUENTE SUCCESSIONE (a_n):

138 $a_n = \left(\sqrt{n}-\lfloor\sqrt{n}\rfloor\right)^{\sqrt{n}}$

VENERDÌ 11 NOVEMBRE IL VIDEO CON LA SOLUZIONE DI QUESTO PROBLEMA E LA PROPOSTA DI UN NUOVO PROBLEMA DA RISOLVERE

SOLUZIONI

44 $$\lim_{n\to+\infty} \frac{n^5 2^n + n^{10}}{n^2+3^n} =$$

$$= \lim_{n\to+\infty} \frac{\overbrace{n^5\cdot 2^n}^{0}\overbrace{\left(1+\frac{n^{10}}{n^5\cdot 2^n}\right)}^{\downarrow 1}}{3^n\left(\frac{n^2}{3^n}+1\right)} = 0$$

$$\frac{n^{10}}{n^5\cdot 2^n} = \frac{n^5}{2^n} \to 0$$

$$\frac{n^2}{3^n} \to 0$$

$$\frac{n^5\cdot 2^n}{3^n} = \frac{n^5}{\left(\frac{3}{2}\right)^n} \to 0$$

$$\lim_{n\to+\infty} \frac{n^5\cdot 2^n + o(n^5\cdot 2^n)}{o(3^n)+3^n} = \lim_{n\to+\infty} \frac{n^5\cdot 2^n}{3^n} = 0$$

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$$\lim_{n\to+\infty} \frac{e^{n^2+n}}{e^{n^2}} = \lim_{n\to+\infty} e^n = +\infty$$

NO

NO

$$\lim_{n\to+\infty} \frac{e^{n^2+\cancel{o(n^2)}}}{e^{n^2}} = 1$$

$$\boxed{51}\ \lim_{n\to+\infty} \frac{\sqrt{n^{n+1}}+2^{n^2}}{(2n)!} =$$

$$\frac{2^{n^2}}{n^{\frac{n+1}{2}}} > \frac{2^{n^2}}{n^n} = \Big(\underbrace{\frac{2^n}{n}}_{+\infty}\Big)^n > \frac{2^n}{n} \to +\infty$$

$$= \lim_{n\to+\infty} \frac{o(2^{n^2})+2^{n^2}}{(2n)!} =$$

$$= \lim_{n\to+\infty} \frac{2^{n^2}}{(2n)!} = +\infty$$

$$(2n)! < (2n)^{2n}$$

$$m! < m^m$$

$$\boxed{\frac{2^{n^2}}{(2n)!}} > \frac{2^{n^2}}{(2n)^{2n}} = \Big(\underbrace{\frac{2^n}{4n^2}}_{+\infty}\Big)^n > \boxed{\frac{2^n}{4n^2}} \to +\infty$$

$$(+\infty)^{+\infty}$$

$$\boxed{\boxed{94}\ \lim_{n\to+\infty} \sqrt[n]{n} = 1}$$

$$n^{\frac{1}{n}} \qquad (+\infty)^0$$

$$n^{\frac{1}{n}} = e^{\ln\left(n^{\frac{1}{n}}\right)} = e^{\frac{\ln n}{n}} \to e^0 = 1$$

$$\frac{\ln n}{n} \to 0$$

$$\boxed{A_n \simeq a_n} \qquad \boxed{B_n \simeq b_n}$$

$$\lim_{n\to+\infty} \frac{A_n}{B_n} = \lim_{n\to+\infty} \underbrace{\frac{A_n}{a_n}}_{1} \cdot \frac{a_n}{b_n} \cdot \underbrace{\frac{b_n}{B_n}}_{1} = \lim_{n\to+\infty} \frac{a_n}{b_n}$$

74 $\lim_{n\to+\infty}\left(1+\frac{1}{n+2}\right)^n =$

$$= \lim_{n\to+\infty} \underbrace{\boxed{\left(1+\frac{1}{n+2}\right)^{n+2}}}_{\downarrow\, e} \cdot \frac{1}{\underbrace{\left(1+\frac{1}{n+2}\right)}_{\downarrow\, 1}\underbrace{\left(1+\frac{1}{n+2}\right)}_{\downarrow\, 1}} =$$

$$= e$$

$a_n \approx b_n$

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69 $\lim_{n\to+\infty}\left(1-\frac{3}{n}\right)^n =$

$$= \lim_{n\to+\infty}\left(\frac{n-3}{n}\right)^n =$$

$$= \lim_{n\to+\infty} \frac{1}{\left(\frac{n-3+3}{n-3}\right)^n} = \lim_{n\to+\infty} \frac{1}{\left(1+\frac{3}{n-3}\right)^n}$$

$$\lim_{n\to+\infty} \frac{1}{\left(\underbrace{\left(1+\frac{1}{\frac{n-3}{3}}\right)^{\frac{n-3}{3}}}_{\downarrow\, e}\right)^{\frac{3n}{n-3}\to 3}} = \frac{1}{e^3}$$

$$\boxed{\left(1+\frac{x}{n}\right)^n \to e^x}$$

$$\left(1+\frac{1}{n}\right)^n \to e$$

$$\to \boxed{\left(1+\frac{1}{a_n}\right)^{a_n} \to e} \qquad \text{SE } \boxed{a_n \to \begin{matrix}+\infty\\-\infty\end{matrix}}$$

$$\uparrow\ \uparrow$$

$$\lim_{n\to+\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n\to+\infty}\left(\underbrace{\left(1+\frac{1}{\frac{n}{x}}\right)^{\frac{n}{x}}}_{\downarrow\, e}\right)^{x\to x} = e^x$$

$$\boxed{a_n\to+\infty}$$

$$\left(1+\frac{1}{k}\right)^{k+1} \to e$$

$$\underbrace{\left(1+\frac{1}{\lfloor a_n\rfloor+1}\right)^{\lfloor a_n\rfloor}}_{\downarrow\, e} < \boxed{\left(1+\frac{1}{a_n}\right)^{a_n}} < \underbrace{\left(1+\frac{1}{\lfloor a_n\rfloor}\right)^{\lfloor a_n\rfloor+1}}_{\downarrow\, e}$$

$$\left(1+\frac{1}{k+1}\right)^{k} \to e \ ?$$

$$\left[\underbrace{\left(1+\frac{1}{k+1}\right)^{k+1}}_{\downarrow \atop e} \cdot \underbrace{\frac{1}{1+\frac{1}{k+1}}}_{\downarrow \atop 1}\right] \to e$$

$a_n \to -\infty$

$$\left(1+\frac{1}{a_n}\right)^{a_n} \to e$$

$$\| $$

$$\left(\frac{a_n+1}{a_n}\right)^{a_n} = \left(\frac{-a_n}{-a_n-1}\right)^{-a_n} =$$

$$= \left(\frac{-a_n-1+1}{-a_n-1}\right)^{-a_n} =$$

$b_n = -a_n - 1 \to +\infty$

$$= \left(1+\frac{1}{-a_n-1}\right)^{-a_n-1} \cdot \left(1+\frac{1}{-a_n-1}\right)$$

$$= \underbrace{\left(1+\frac{1}{b_n}\right)^{b_n}}_{\downarrow \atop e} \cdot \underbrace{\left(1+\frac{1}{b_n}\right)}_{\downarrow \atop 1}$$

81 $$\lim_{n\to+\infty}\left(\frac{\overbrace{n+9}^{n+5+4}}{n+5}\right)^{3n} =$$

$$= \lim_{n\to+\infty}\left(1+\frac{4}{n+5}\right)^{3n} =$$

$$= \lim_{n\to+\infty}\Bigg(\underbrace{\left(1+\frac{1}{\frac{n+5}{4}}\right)^{\frac{n+5}{4}}}_{\downarrow\, e}\Bigg)^{\frac{12n}{n+5}} = e^{12}$$

$$\frac{12n}{n}\to 12$$

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86 $$\lim_{n\to+\infty}\left(\frac{n^2-n-6}{n^2-n+2}\right)^{\frac{n^4+n+3}{n^2-1}} =$$

$$= \lim_{n\to+\infty}\left(\frac{(n^2-n+2)-8}{(n^2-n+2)}\right)^{\boxed{\frac{n^4+n+3}{n^2-1}}} =$$

$$= \lim_{n\to+\infty}\Bigg(\underbrace{\left(1+\frac{1}{-\frac{n^2-n+2}{8}}\right)^{-\frac{n^2-n+2}{8}}}_{\downarrow\, e}\Bigg)^{-\frac{8(n^4+n+3)}{(n^2-n+2)(n^2-1)}} = \frac{1}{e^8}$$

$$= -\frac{\boxed{8n^4}+o(8n^4)}{\boxed{n^4}+o(n^4)}$$

$$\underbrace{o(a_n)+o(a_n)}=o(a_n)$$

$$\rightarrow \underbrace{o(a_n)-o(a_n)}=o(a_n)$$

$$\left.\begin{array}{l}\boxed{b_n=o(a_n)}\\ \boxed{c_n=o(a_n)}\end{array}\right\} \Rightarrow \boxed{b_n\mp c_n=o(a_n)}$$

$$\boxed{\begin{array}{l}\frac{b_n}{a_n}\to 0\\ \frac{c_n}{a_n}\to 0\end{array}}\ ?$$

$$\frac{b_n\mp c_n}{a_n}\to 0 \quad ?$$

$$\parallel$$

$$\frac{b_n}{a_n}\mp\frac{c_n}{a_n}\to 0\pm 0=0$$

$+7-13$

84 $\displaystyle\lim_{n\to+\infty}\left(\frac{n^2+n-6}{n^2+n+7}\right)^n=$

$$=\lim_{n\to+\infty}\left(\underbrace{\left(1+\frac{1}{-\frac{n^2+n+7}{13}}\right)^{-\frac{n^2+n+7}{13}}}_{\downarrow\, e}\right)^{\frac{-13n}{n^2+n+7}}=e^{0}$$

$$=\lim_{n\to+\infty}\left(1+\frac{1}{-\frac{n^2+n+7}{13}}\right)^{n}=1$$

$b_n = o(a_n)$

$$\lim_{n\to+\infty}\left(1+\frac{1}{a_n}\right)^{b_n} = 1$$

$$= \lim_{n\to+\infty}\Bigg(\underbrace{\left(1+\frac{1}{a_n}\right)^{a_n}}_{e}\Bigg)^{\frac{b_n}{a_n}\to 0} = e^0 = 1$$

$a_n = o(b_n)$ $\quad \lim_{n\to+\infty}\left(1+\frac{1}{a_n}\right)^{b_n} =$

$a_n \to +\infty$, $b_n \to +\infty$ ①

$a_n \to +\infty$, $b_n \to -\infty$ ②

$$\lim_{n\to+\infty}\Bigg(\underbrace{\left(1+\frac{1}{a_n}\right)^{a_n}}_{e}\Bigg)^{\frac{b_n}{a_n}\to\pm\infty} = \begin{cases} e^{+\infty} = +\infty & ① \\ e^{-\infty} = 0 & ② \end{cases}$$

108 $\displaystyle\lim_{n\to+\infty}\frac{\overbrace{(n+1)^{n+\frac{1}{n}}}+\overbrace{n!}^{o(\cdot)}}{(n+2)^n-n^n}=$

$$=\lim_{n\to+\infty}\frac{(n+1)^n\cdot\sqrt[n]{n+1}}{(n+2)^n-n^n}=$$

$$=\lim_{n\to+\infty}\frac{\overbrace{\left(1+\frac{1}{n}\right)^n}^{\to e}\cdot\overbrace{\sqrt[n]{n+1}}^{\to 1}}{\underbrace{\left(1+\frac{2}{n}\right)^n}_{\to e^2}-\underbrace{1}_{\to 1}}=\frac{e}{e^2-1}$$

$n!=o(n^n)$

$$\boxed{n^n}$$
$$1$$

$$0\le\boxed{\frac{n!}{///n}}=\frac{n!}{n^n}\cdot\frac{n^n}{\text{\~{}\~{}\~{}}}\le$$

<1

$$\le\boxed{\frac{n!}{n^n}}\to 0$$

$$\overbrace{\sqrt[n]{n+1}\to 1}\qquad\overbrace{\sqrt[n]{n}\to 1}$$

$$\underbrace{\sqrt[n]{n}}_{\to 1}<\underbrace{\sqrt[n]{n+1}}_{\to 1}<\sqrt[n]{n+n}=\sqrt[n]{2n}=\underbrace{\sqrt[n]{2}}_{\to 1}\cdot\underbrace{\sqrt[n]{n}}_{\to 1}$$

$$2^{\frac{1}{n}}\to 2^0=1$$

124 $\displaystyle\lim_{n\to+\infty} \frac{(n!)!}{n^{n^n}} =$

$$= \lim_{n\to+\infty} \frac{(n!)!}{(n^n)^{n^{n-1}}} = 0$$

$$0 \leq \frac{(n!)!}{(n^n)^{n^{n-1}}} < \frac{(n!)!}{(n!)^{n!}} \to 0$$

$\downarrow$ 0

$(k = n!)$

$$\frac{k!}{k^k} \to 0$$

$$n! = o(n^{n-1})$$

$$\frac{n!}{n^{n-1}} \quad \frac{n\cdot(n-1)\cdots 3\cdot 2\cdot 1}{n\cdot n \cdots\cdots n\cdot n}$$

$\downarrow$ 0 $\quad$ $\downarrow$ (≤ 1)

$$n^{n^n} = (n^n)^{n^{n-1}}$$

$n! \cdot n!$

119 $$\lim_{n\to+\infty} \underbrace{\frac{(n!)^2}{n^n}}_{a_n} = +\infty$$

$k = \frac{n}{2}$

$(n-k+1)\cdot k$

$$\frac{(n\cdot 1)\cdot((n-1)\cdot 2)\cdot((n-2)\cdot 3)\cdots(1\cdot n)}{n \quad n \quad n \quad n}$$

$$\frac{\boxed{\left(n-\frac{n}{2}+1\right)\frac{n}{2}}}{n}$$

$$\boxed{\frac{a_{n+1}}{a_n}} = \frac{((n+1)!)^2}{(n+1)^{n+1}} \cdot \frac{n^n}{(\cancel{n!})^2} =$$

$l > k l$

$$a_{n+1} > 2 a_n \to 2l = \frac{(n+1)^2}{\cancel{(n+1)}} \cdot \left(\frac{n}{n+1}\right)^n = \underbrace{(n+1)}^{\to +\infty} \cdot \frac{1}{\underbrace{\left(1+\frac{1}{n}\right)^n}_{\to \frac{1}{e}}}$$

$a_{n+1} \to l$, $a_n \to l$