Esercitazioni di Analisi 1

Exe 8

8 Novembre 2022

PROBLEMI PROPOSTI

1) DATA $a_n = (-1)^n\left(1+\frac{1}{n}\right) + \frac{\sin n}{2^n}$ CALCOLARE $\liminf_{n\to+\infty} a_n$ E $\limsup_{n\to+\infty} a_n$.

2) $\boxed{\text{TEOREMA}}$ DATA (a_n) (LIMITATA) SIANO ℓ_1, ℓ_2 ED ℓ_3 DATI DA:

$\ell_1 = \lim_{n\to+\infty} A_n$ DOVE $A_n = \sup_{n} a_k$

$\ell_2 = \inf \mathcal{M}$ DOVE $\mathcal{M} = \{\lambda \in \mathbb{R} \mid \text{DEF. IN } n \ a_n \leq \lambda\} =$ $\boxed{\text{INSIEME DEI MAGGIORANTI DEFINITIVI DI } (a_n)}$

$\ell_3 = \text{MAX } \mathcal{L}$ DOVE $\mathcal{L} = \{\ell \in \mathbb{R} \mid \exists (a_{n_k}) \to \ell\} =$ $\boxed{\text{INSIEME DEI PUNTI LIMITE DI } (a_n)}$

ALLORA $\ell_1 = \ell_2 = \ell_3$

3) **Esercizio 2 [6 punti]** (DAL I° APPELLO ESTIVO A.A. 21/22

Data $a_n = (1+\sin n)^n$, calcolare $\liminf_{n\to+\infty} a_n$ e $\limsup_{n\to+\infty} a_n$.

VARIAZIONI SUL PROB. 1

3.1) $a_n = n\cdot(1+\sin n)^n$

3.2) $a_n = \frac{1}{n}(1+\sin n)^n$

3.3) $a_n = \left(\frac{2}{3}\right)^n\cdot(1+\sin n)^n$

3.4) $a_n = 100^n(1+\sin n)^n$

TROVARE $\limsup a_n$ E $\liminf a_n$ NEI SEGUENTI CASI

4) ⌊135 $a_n = \left(1+\frac{\sin\frac{n\pi}{3}}{n}\right)^n$

5) ⌊136 $a_n = \left(1+\frac{\sin n}{n}\right)^n$

6) ⌊139 $a_n = \sqrt{n}\,(\sin\sqrt{n})^2$

PROBLEMA DEL VENERDÌ

TROVARE TUTTI I PUNTI LIMITE DI:

138 $a_n = \left(\sqrt{n} - \lfloor \sqrt{n} \rfloor\right)^{\sqrt{n}}$

... RIMASTI DALLA VOLTA SCORSA

7) 114 $\lim_{n \to +\infty} \left((n+9)^{100} + n^{98} \ln(n!) - n^{100}\right) \cdot \left(\sqrt{n^{199}+1} - \sqrt{n^{199}-1}\right)$

8) 117 $\lim_{n \to +\infty} \frac{(n!)^2 \cdot 5^n}{(2n)!}$

9) 118 $\lim_{n \to +\infty} \frac{(n!)^2 \cdot 4^n}{(2n)!}$

SOLUZIONI

$$(n+9)^{100} - n^{100} + \boxed{n^{98}\ln(n!)}$$

$$\cancel{n^{100}} + 100n^{99} + \dots - \cancel{n^{100}} + n^{98}\ln(n!)$$

$$\frac{n^{98}\ln(n!)}{n^{99}} = \boxed{\frac{\ln(n!)}{n}} \longrightarrow +\infty$$

$\forall M > 0$ DEF. IN n $\frac{\ln(n!)}{n} > M$??

DEF. IN n $M^n < n!$ $\ln(n!) > n\ln(M)$

$$\boxed{(e^M)^n < n!} \qquad \frac{\ln(n!)}{n} > \ln(M)$$

$$n \ln((e^M)^n) < \ln(n!)$$

$$nM < \ln(n!) \qquad \boxed{\frac{\ln(n!)}{n} > M}$$

$$0 \le \frac{n^{98}\ln(n!)}{n^{100}} = \frac{\ln(n!)}{n^2} \le \frac{\ln(n^n)}{n^2} = \frac{\cancel{n}\ln n}{n^{\cancel{2}}} = \frac{\ln n}{n} \to 0$$

$$\lim_{n\to+\infty} n^{98}\cdot\ln(n!)\cdot\left(\frac{\sqrt{n^{199}+1}-\sqrt{n^{199}-1}}{1}\right) =$$

$$= \lim_{n\to+\infty} \frac{2\cdot n^{98}\cdot\ln(n!)}{\sqrt{n^{199}+1}+\sqrt{n^{199}-1}} = 0$$

$$\frac{199}{2} = 99 + \frac{1}{2}$$

$$n^{\frac{199}{2}} = n^{99+\frac{1}{2}}$$

$$0 \le \bullet \le \frac{2n^{98}\ln(n^n)}{n^{\frac{199}{2}}} = \boxed{\frac{2\cancel{n^{99}}\ln n}{\cancel{n^{99}}\cdot\sqrt{n}}}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$0 \qquad 0 \qquad 0$$

$$n^{\frac{199}{2}} = n^{99+\frac{1}{2}} = n^{99} \cdot n^{\frac{1}{2}} = \boxed{n^{99} \cdot \sqrt{n}}$$

$a_n > 0 \quad \forall n$

$a_n \to +\infty$?

$a_n \to 0$?

$\boxed{\exists \lambda > 1 \text{ t.c. } \underbrace{\text{DEF. IN } n}_{} \quad \frac{a_{n+1}}{a_n} > \lambda} \Rightarrow a_n \to +\infty$

$\boxed{\exists \lambda \in (0,1) \text{ T.C. DEF. IN } n \quad \frac{a_{n+1}}{a_n} < \lambda} \Rightarrow a_n \to 0$

$0 < \lambda < 1$

$\forall k$

$\forall n \geq n_0 \quad \boxed{\frac{a_{n+1}}{a_n} > \lambda} \qquad a_{n_0+k} > \lambda^k \boxed{a_{n_0}}$

$\lambda^k \to +\infty$

$\underbrace{a_{n_0+1} > \lambda a_{n_0}}_{}$

$a_{n_0+2} > \lambda a_{n_0+1} > \lambda^2 a_{n_0}$

$\vdots$

$\boxed{\underbrace{a_{n_0+k} > \lambda^k a_{n_0}}_{}}$

$a_{n_0+k+1} > \lambda \overbrace{a_{n_0+k}} > \lambda \cdot \lambda^k \cdot a_{n_0} = \lambda^{k+1} a_{n_0}$

$$0 \le a_{n_0+k} < \underbrace{a_{n_0}}\cdot \lambda^k$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$0 \quad 0 \quad 0$$

$$\downarrow$$

$$0$$

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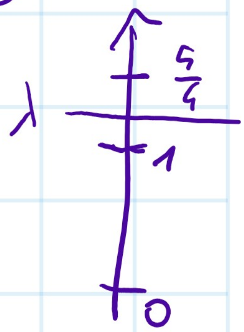
117 $\displaystyle\lim_{n\to+\infty} \frac{(n!)^2\cdot 5^n}{(2n)!} = +\infty$

a_n

118 $\displaystyle\lim_{n\to+\infty} \underbrace{\frac{(n!)^2\cdot 4^n}{(2n)!}}_{b_n}$

$$\boxed{\frac{a_{n+1}}{a_n}} = \frac{\overbrace{((n+1)!)^2}^{(n+1)^2}\cdot \cancel{5^{n+1}}}{\underbrace{\cancel{(2n+2)!}}_{(2n+2)(2n+1)}} \cdot \frac{\cancel{(2n)!}^{\,1}}{\underbrace{\cancel{(n!)^2}}_{1}\cdot\cancel{5^n}} =$$



$$= \frac{(n+1)^{\cancel{2}}\;5}{2\,\cancel{(2n+2)}\underbrace{(2n+1)}_{2\left(n+\frac12\right)}} = \frac{5}{4}\boxed{\frac{(n+1)}{\left(n+\frac12\right)}}_{\downarrow\, 1} \longrightarrow \frac{5}{4} > 1$$

$$\boxed{\frac{b_{n+1}}{b_n}} = \cdots = \frac{\cancel{4}(n+1)}{\cancel{4}\left(n+\frac12\right)} = \frac{2n+2}{2n+1} = \boxed{1+\frac{1}{2n+1}}$$

$\boxed{c_n \to +\infty}$

$\boxed{\text{def. } i_n}$ $\quad \dfrac{b_{n+1}}{b_n} > \dfrac{c_{n+1}}{c_n} \implies b_n \to +\infty$

$\exists n_0$ t.q. $\boxed{n > n_0}$

$$\boxed{\frac{b_{n+1}}{b_n} > \frac{c_{n+1}}{c_n}} \qquad \frac{b_{n_0+1}}{b_{n_0}} > \frac{c_{n_0+1}}{c_{n_0}}$$

$$\frac{b_{n_0+k}}{b_{n_0}} > \frac{c_{n_0+k}}{c_{n_0}} \quad (?)$$

$$\boxed{\frac{b_{n_0+k}}{b_{n_0+k-1}}} \cdot \boxed{\frac{b_{n_0+k-1}}{b_{n_0+k-2}}} \cdot \ldots \cdot \frac{b_{n_0+2}}{b_{n_0+1}} \cdot \boxed{\frac{b_{n_0+1}}{b_{n_0}}} > \boxed{\frac{c_{n_0+k}}{c_{n_0+k-1}}} \cdot \boxed{\frac{c_{n_0+k-1}}{c_{n_0+k-2}}} \cdots \boxed{\frac{c_{n_0+1}}{c_{n_0}}}$$

$$\frac{b_{n_0+k}}{b_{n_0}} > \frac{c_{n_0+k}}{c_{n_0}}$$

$$\underbrace{\boxed{b_{n_0+k}}}_{\downarrow} > \boxed{\frac{b_{n_0}}{c_0}} \cdot \underbrace{\boxed{c_{n_0+k}}}_{\downarrow\, +\infty}$$

$$\boxed{\frac{b_{n+1}}{b_n}} = \boxed{1 + \frac{1}{2n+1}} \qquad \boxed{c_n = \sqrt{n+1}}$$

$$\frac{c_{n+1}}{c_n} = \frac{\sqrt{n+1}}{\sqrt{n}} =$$

$$\boxed{\frac{c_{n+1}}{c_n}} = \boxed{\sqrt{\frac{n+2}{n+1}}} = \sqrt{1 + \frac{1}{n+1}}$$

$$= \sqrt{1 + \frac{1}{n}} \approx 1 + \frac{1}{2n}$$

$$\left(1+\frac{1}{2n+1}\right)^2 \overset{\text{SI}}{>} \left(\sqrt{1+\frac{1}{n+1}}\right)^2 \quad (?)$$

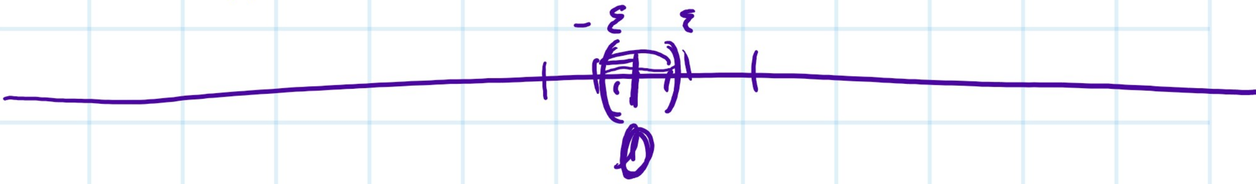
$$1+\frac{2}{2n+1}+\left(\frac{1}{2n+1}\right)^2 \overset{(?)}{>} 1+\frac{1}{n+1}$$

$$\Updownarrow$$

$$\frac{1}{n+\frac{1}{2}}$$

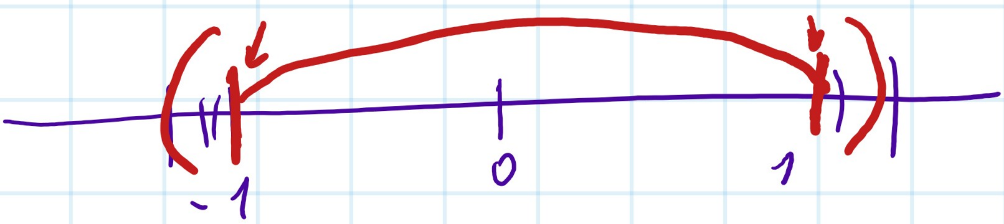
$$\boxed{(a_n) \qquad \lim_{n\to+\infty} \boxed{\left(\sup_{k\geq n} a_k\right)}}$$

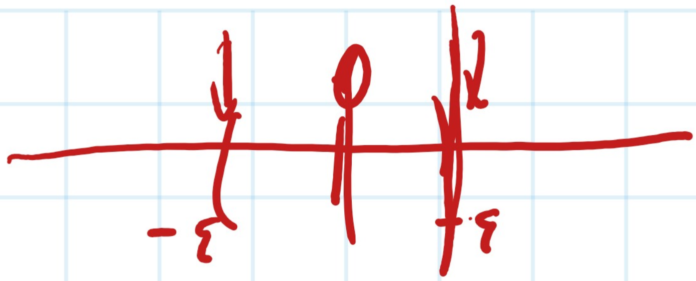
$$(-1)^n\cdot\frac{1}{n}$$

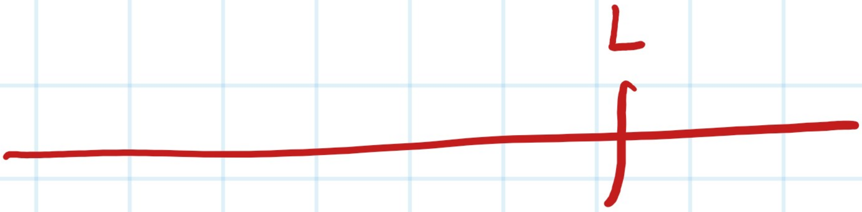


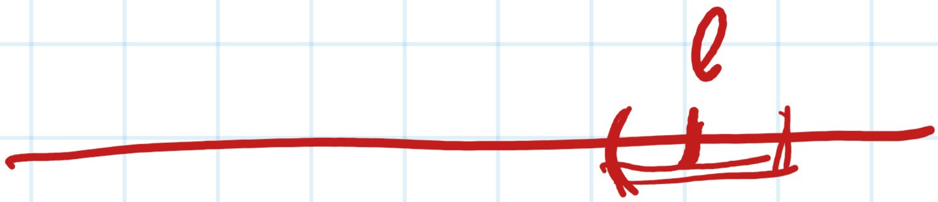
$$(-1)^n\left(1+\frac{1}{n}\right)$$

$$\downarrow$$

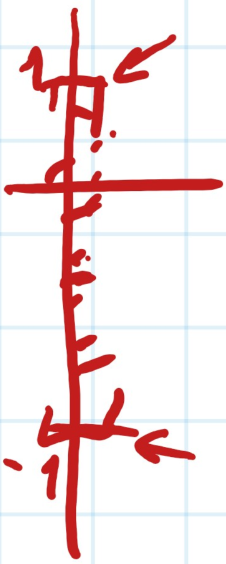


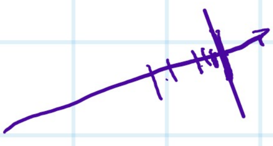




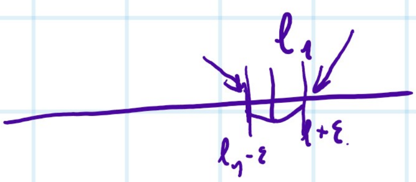


$a_n = \sin n$





A_n



2) $\boxed{\text{TEOREMA}}$ DATA (a_n) (LIMITATA) SIANO ℓ_1, ℓ_2 ED ℓ_3 DATI DA:

$\ell_1 = \lim_{n \to +\infty} \boxed{A_n}$ DOVE $A_n = \sup_{k \geq n} a_k$

$\ell_2 = \inf \mathcal{M}$ DOVE $\mathcal{M} = \{ \lambda \in \mathbb{R} \mid \text{DEF. IN } n \ \ a_n \leq \lambda \} =$ INSIEME DEI MAGGIORANTI DEFINITIVI DI (a_n)

$\longrightarrow \ell_3 = \max \mathcal{L}$ DOVE $\mathcal{L} = \{ \ell \in \mathbb{R} \mid \exists (a_{n_k}) \to \ell \} =$ INSIEME DEI PUNTI LIMITE DI (a_n)

ALLORA $\ell_1 = \ell_2 = \ell_3$

$\boxed{\text{DIM}}$

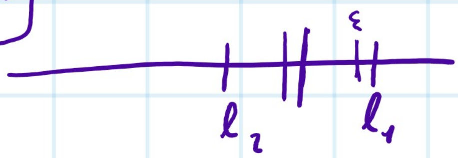
OSS. $\forall n \ \ A_n \in \mathcal{M}$ $\qquad \{ A_n \mid n \in \mathbb{N} \} \subset \mathcal{M}$

$A_n = \sup_{k \geq n} a_k$

$$\ell_1 = \lim_{n \to +\infty} A_n = \inf_{n \in \mathbb{N}} A_n \geq \inf \mathcal{M} = \ell_2$$

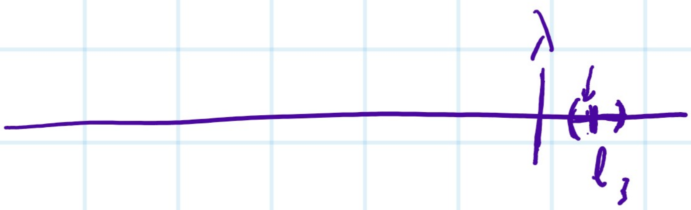
$\ell_1 \geq \ell_2$

$\boxed{\ell_1 \leq \ell_2}$



$\boxed{\ell_3} \leq \lambda \in \mathcal{M} \implies \ell_3$ È MINORANTE PER $\mathcal{M} \implies \ell_3 \leq \underset{\ell_1}{\overset{}{\ell_2}}$ ($\ell_1 = \ell_2$)

$\ell_3 > \lambda$

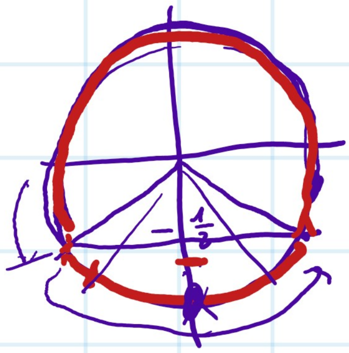


$\ell_3 \geq \ell_2$
$\quad \geq \ell_1$

Esercizio 2 [6 punti]

Data $a_n = (1+\sin n)^n$, calcolare $\liminf_{n\to+\infty} a_n$ e $\limsup_{n\to+\infty} a_n$.

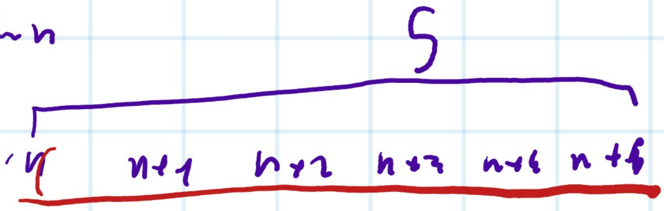
$$(1+\sin n)^n$$



$-\frac{\pi}{6}, \quad \frac{7}{6}\pi$

$\frac{8}{6}\pi \quad \boxed{\frac{4}{3}\pi}$

$\sin n$



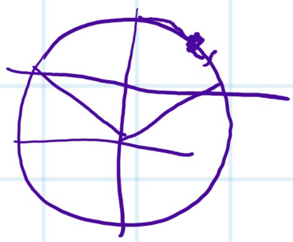
FREQ. IN n $\quad \sin n < -\frac{1}{2}$

$\exists\, (n_k)$ t.c. $\quad \sin n_k < -\frac{1}{2} \quad \forall k$

$$0 < (1+\sin n_k)^{n_k} < \left(\frac{1}{2}\right)^{n_k}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$0 \qquad 0 \qquad 0$$



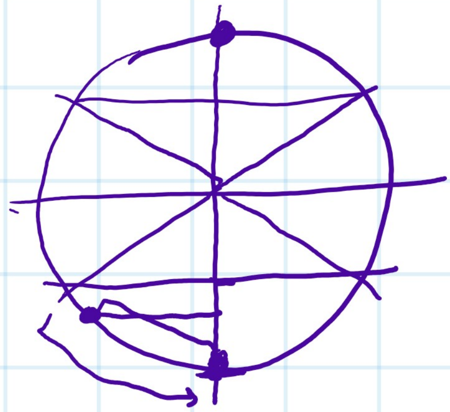
$n_k = \lfloor 2k\pi + \frac{3}{2}\pi \rfloor$

$\lfloor 2k\pi + \frac{\pi}{2} \rfloor$

n

$$\boxed{\left(1+\sin n_k\right)^{n_k}} > \left(\frac{3}{2}\right)^{n_k}$$

$$2k\pi + \frac{3}{2}\pi - 1 < \left[2n\pi + \frac{3}{2}\pi\right] < 2k\pi + \frac{3}{2}\pi$$



$$\boxed{\frac{\pi}{3} > 1}$$