Esercitazioni di Analisi 1

Exe 10

15 Novembre 2022

PROBLEMI PROPOSTI

1) $\lim_{x\to 0} \dfrac{\tan 2x - \sin 2x}{\ln(\cos x)\cdot \sin(\sin x)}$

2) $\lim_{x\to 0} \dfrac{\tan x^2 - \sin x^2}{(\tan^2 x - \sin^2 x)\cdot x^\alpha}$

3) $\lim_{x\to +\infty} \dfrac{x}{\cos(x^2+1)}$

4) $\lim_{x\to 0} x^x$

5) $\lim_{x\to 0} \dfrac{e^{2x^2} - \cos 3x}{\ln\sqrt{1+x^2}}$

6) $\lim_{x\to 0} \dfrac{\cos(\ln(\cos x)) - 1}{x^\alpha}$

7) $\lim_{x\to 0} \dfrac{\sin x + \sin 2x + \dots + \sin 100x}{\sin(\sin(\sin(x)))}$

8) $\lim_{x\to 0} \dfrac{2\sin x - \sin 2x}{x^\alpha}$

9) $\lim_{x\to +\infty} \ln(1+e^x)\cdot \sin\frac{1}{x}$

10) $\lim_{x\to \frac{\pi}{2}} \dfrac{\ln(\sin x)}{\cos x}$

11) $\lim_{x\to +\infty} \dfrac{\sqrt{\frac{x+2}{x+1}} - \sqrt{\frac{x+1}{x+2}}}{\ln(x+1) - \ln x}$

12) $\lim_{x\to +\infty} \dfrac{\left(\frac{\pi}{2} - \arctan x\right)\ln(1+x)}{\ln(1+x^x)}$

13) $\lim_{n\to +\infty} \sqrt{n}\cdot \sin\left(\sqrt{n+1} - \sqrt{n}\right)$

14) $\lim_{n\to +\infty} \dfrac{\tan\left(\frac{1}{n!}\right) - \sin\left(\frac{1}{n!}\right)}{1 - \cos\left(\frac{1}{n^n}\right)}$

1

$$\lim_{x\to 0} \frac{\overbrace{\tan 2x - \sin 2x}^{4x^3}}{\underbrace{(\ln(\cos x))}_{-\frac{1}{2}x^2} \cdot \underbrace{(\sin(\sin x))}_{x}} =$$

$\sin(\sin x) \approx \sin x \approx x$

$\tan 2x - \sin 2x \approx \frac{1}{2}(2x)^3 = 4x^3$

$\ln(1+(\cos x - 1)) \approx \cos x - 1 \approx -\frac{1}{2}x^2$

$$= \lim_{x\to 0} \boxed{\frac{\tan 2x - \sin 2x}{\ln(1+(\cos x-1))\,\sin(\sin x)}} =$$

$$\frac{\ln(1+y)}{y} \to 1$$

$$= \lim_{x\to 0} \underbrace{\frac{\tan 2x - \sin 2x}{(2x)^3}}_{\downarrow\ \frac{1}{2}} \cdot \underbrace{\frac{(\cos x - 1)}{\ln(1+(\cos x - 1))}}_{\downarrow\ 1} \cdot \underbrace{\frac{\sin x}{\sin(\sin x)}}_{\downarrow\ 1} \cdot \underbrace{\frac{(2\cancel{x})^3}{\cancel{x}\cancel{x^2}}}_{\downarrow\ 8} \cdot \underbrace{\frac{x^2}{\cos x - 1}}_{\downarrow\ -2} \cdot \underbrace{\frac{x}{\sin x}}_{\downarrow\ 1}$$

$$= -8$$

$$\lim_{x\to 0} \underbrace{\boxed{\frac{\tan 2x - \sin 2x}{\frac{1}{2}(2x)^3}}}_{\downarrow\ 1} \frac{\frac{1}{2}(2x)^3}{\ln(\cos x)\cdot\sin(\sin x)}$$

$f(x) \to 0$ per $x \to x_0$

$$\lim_{x\to x_0} \frac{1-\cos(f(x))}{(f(x))^2} = \lim_{y\to 0} \frac{1-\cos y}{y} = \frac{1}{2}$$

$y = f(x)$

$$\lim_{x \to x_0} \frac{1-\cos(f(x))}{\frac{1}{2}(f(x))^2} = 1 \qquad \boxed{\text{SE } f(x) \to 0 \text{ PER } x \to x_0}$$

$$\boxed{1-\cos(f(x)) \approx \frac{1}{2}(f(x))^2}$$

$x \to x_0$
$$\ln(1+f(x)) \approx f(x)$$

$$\boxed{e^{f(x)}-1 \approx f(x)}$$

2

$$\lim_{x \to 0^+} \frac{\tan(x^2)-\sin(x^2)}{\underbrace{(\tan^2 x - \sin^2 x)}_{\underbrace{(\tan x - \sin x)}(\tan x + \sin x)} \cdot x^\alpha} = \lim_{x \to 0^+} \frac{\frac{1}{2}(x^2)^3}{\frac{1}{2}x^3 \cdot 2x \cdot x^\alpha} = \lim_{x \to 0^+} \frac{x^6}{2x^6 \cdot x^\alpha}$$

$$= \lim_{x \to 0} \frac{x^2}{2x^\alpha} =$$

$$\frac{\tan x + \sin x}{2x} \to 1$$

$$\begin{cases} +\infty & \alpha > 2 \\ \frac{1}{2} & \alpha = 2 \\ 0 & 0 < \alpha < 2 \end{cases}$$

3

$$\lim_{x \to +\infty} \underbrace{\left[\frac{x}{\cos(x^2+1)} \right]}^{f(x)}$$

$$x^2+1 = 2k\pi$$

$$x^2 = 2k\pi - 1$$

$$x = \boxed{\sqrt{2k\pi - 1} = a_k}$$

$$f(a_k) = \frac{a_k}{\cos((a_k)^2+1)} = \frac{\sqrt{2k\pi - 1}}{\cos 2k\pi} = \sqrt{2k\pi - 1} \to +\infty$$

$$x^2+1 = (2k+1)\pi$$

$$x = \boxed{\sqrt{(2k+1)\pi - 1} = b_k}$$

$$f(b_k) = \frac{b_k}{\cos((b_k)^2+1)} = \frac{\sqrt{(2k+1)\pi - 1}}{\cos((2k+1)\pi)} = -\sqrt{(2k+1)\pi - 1} \to -\infty$$

4

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0} e^{\ln(x^x)} = \lim_{x \to 0} e^{x \ln x} = e^0 = 1$$

$x \to +\infty$

$$\frac{\ln x}{x} \to 0 \quad \xrightarrow{\forall\, a_n \to +\infty} \quad \frac{\ln |a_n|}{a_n} \to 0$$

$x \to 0^+$ $\quad \frac{1}{x} \to +\infty$

$$x \ln x = -x \ln x^{-1} = -\frac{\ln\left(\frac{1}{x}\right)}{\frac{1}{x}} \to 0$$

$x \ln x \to 0$

$y \to +\infty$

$$-\frac{\ln y}{y} \to 0$$

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5 $$\lim_{x\to 0} \frac{e^{2x^2} - \cos 3x}{\ln\sqrt{1+x^2}} = \lim_{x\to 0} \frac{\left(e^{2x^2}-1\right)+\left(1-\cos 3x\right)}{\frac{1}{2}x^2}$$

$$\ln\sqrt{1+x^2} = \frac{1}{2}\ln\left(1+x^2\right) \simeq \frac{1}{2}x^2$$

$$\left(e^{2x^2}-1\right)+\left(1-\cos 3x\right)$$

$$2x^2 + \frac{1}{2}\cdot 9x^2$$

$$\lim_{x\to 0^+} \frac{\tan x - \sin x}{x^4} = \text{NO!!}$$

$$= \lim_{x\to 0^+} \frac{x - x}{x^4} =$$

$$\lim_{x\to 0^+} \frac{0}{x^4} = 0$$

$$\boxed{\lim_{x\to 0^+} \left(\frac{\tan x - \sin x}{x^3}\right)\left(\frac{1}{x}\right) = +\infty}$$

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SE $f(x) \to 0$ PER $x \to x_0$

$$\boxed{\lim_{x \to x_0} \frac{e^{f(x)}-1}{f(x)} = 1}$$

$$\frac{e^x-1}{x} \to 1$$

$$e^x - 1 \approx x$$

$$\boxed{e^{f(x)} - 1 \approx f(x)}$$

$$\boxed{e^x = 1+x+o(x)}$$

$$e^{f(x)} = 1+f(x)+o(f(x))$$

$$e^x-(1+x) = o(x)$$

$x \to x_0$

$f(x), g(x) \to 0$

$$\frac{f(x)}{g(x)} \to 1$$

$x \to x_0$ $\quad \boxed{f(x)-g(x) = o(f(x))} = o(g(x))$

$$\frac{f(x)-g(x)}{f(x)} \to 0 \ ?$$

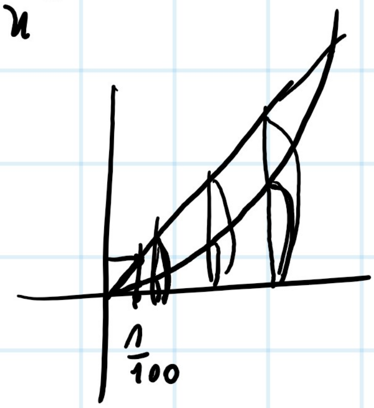
$$\underbrace{\frac{f(x)}{f(x)}}_{1} - \underbrace{\frac{g(x)}{f(x)}}_{1}$$

x

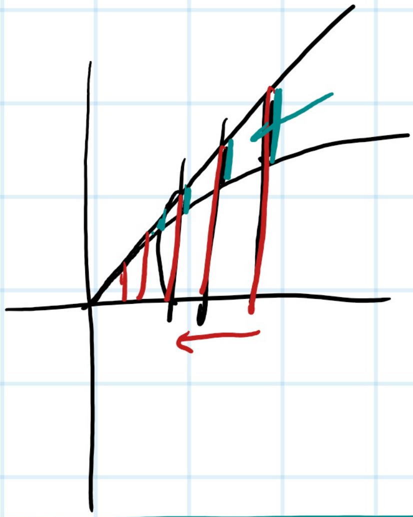
x^2

$x \to 0^+$

$$\frac{x^2}{x} \to 0$$



$\frac{\sin x}{x} \to 1$



$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$

$\ln(1+x) \simeq x$ PER $x \to 0$

$$\ln(1+x) - x = o(x)$$

$$\ln(1+x) = x + o(x)$$

$$\frac{e^x - 1}{x} \to 1$$

$$e^x - 1 \simeq x$$

$(e^x - 1) - x = o(x)$ $\quad e^x = 1 + x + o(x)$

$$e^{x} = 1 + x + o(x)$$

$$\boxed{e^{f(x)}} = \overline{1 + f(x) + o(f(x))}$$

$$\lim_{x\to 0} \frac{\overline{e^{2x^2}} - \cos(3x)}{\ln\sqrt{1+x^2}} =$$

$$= \lim_{x\to 0} \frac{\cancel{1} + 2x^2 + o(x^2) - \left(\cancel{1} - \frac{(3x)^2}{2} + o(x^2)\right)}{\frac{1}{2}x^2} =$$

$$= \lim_{x\to 0} \frac{\overline{\frac{13}{2}x^2 + o(x^2)}}{\frac{1}{2}x^2}$$

$$= \lim_{x\to 0} \frac{\frac{13}{2}x^2}{\frac{1}{2}x^2} = 13$$

$$\frac{\frac{13}{2}x^2 + o(x^2)}{\frac{13}{2}x^2}$$

$$1 + \frac{o(x^2)}{\frac{13}{2}x^2} \to 1$$

$$h(x) = o(x^2)$$

$$k(x) = o(x^2)$$

$$\frac{h(x) - k(x)}{x^2} =$$

$$= \frac{h(x)}{x^2} - \frac{k(x)}{x^2} \to 0 - 0 = 0$$

$$\frac{\overline{1 - \cos x}}{\frac{1}{2}x^2} \to 1 \qquad 1 - \cos x \approx \frac{1}{2}x^2$$

$$-(1 - \cos x) + \frac{1}{2}x^2 = -o\left(\frac{1}{2}x^2\right)$$

$$\boxed{\cos x = 1 - \frac{1}{2}x^2 \underset{+}{-} o\left(\frac{1}{2}x^2\right)}$$

$$h(x) \qquad \overline{o\left(\frac{1}{2}x^2\right)}$$

$$\frac{h(x)}{\frac{1}{2}x^2} \to 0$$

$$\overline{o\left(\frac{1}{2}x^2\right)}$$

$$\boxed{\cos x = 1 - \frac{1}{2}x^2 + \underbrace{o(x^2)}}$$

$$\frac{h(x)}{x^2} \to 0 \qquad \frac{h(x)}{\frac{1}{2}x^2} \to 0$$

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$$\frac{\tan x}{x} \to 1 \qquad \tan x = x + o(x)$$

$$\sin x = x + o(x)$$

$$\lim_{x \to 0} \frac{\tan x - \sin x}{x^4} =$$

$$= \lim_{x \to 0} \frac{x + o(x) - (x + o(x))}{x^4} =$$

$$= \lim_{x \to 0} \frac{o(x) - o(x)}{x^4} = \lim_{x \to 0} \frac{o(x)}{x^4} = ??$$

9 $\lim\limits_{x\to+\infty} \overbrace{\ln(1+e^x)}\cdot \sin\frac{1}{x} = \lim\limits_{x\to 0}\left(x \right)\cdot\frac{1}{x} = \lim\limits_{x\to+\infty} 1 = 1$

$$\ln(1+e^x) = \ln\left(e^x\cdot\left(1+\frac{1}{e^x}\right)\right) =$$

$$= \overbrace{\ln e^x} + \overbrace{\ln\left(1+\frac{1}{e^x}\right)}^{\to 0} =$$

$$= x + o(x) \approx x$$

11

$$\lim_{x\to+\infty} \frac{\sqrt{\frac{x+2}{x+1}}-\sqrt{\frac{x+1}{x+2}}}{\ln(x+1)-\ln x} = \lim \frac{\qquad}{\ln\left(1+\frac{1}{x}\right)}$$

$$\boxed{\lim_{x\to 0} \frac{\sqrt{1+x}-1}{\frac{1}{2}x} = 1}$$

$$\sqrt{\frac{x+2}{x+1}} = \sqrt{\frac{x+1+1}{x+1}} = \sqrt{1+\frac{1}{x+1}}$$

$$\to \sqrt{1+x}-1 \sim \frac{1}{2}x$$

$x \to +\infty$

$$\to \sqrt{1+x} = 1+\frac{1}{2}x+o(x)$$

$$\sqrt{1+\frac{1}{x+1}} = 1+\frac{1}{2}\cdot\frac{1}{x+1}+o\left(\frac{1}{x+1}\right)$$

$f(x)\to 0$

$$\sqrt{1+f(x)} = 1+\frac{1}{2}f(x)+o(f(x))$$

$$\sqrt{\frac{x+1}{x+2}} = \sqrt{\frac{x+2-1}{x+2}} = \sqrt{1+\left(-\frac{1}{x+2}\right)} = 1+\frac{1}{2}\left(-\frac{1}{x+2}\right)+o\left(-\frac{1}{x+2}\right)$$

$$\ldots = \lim_{x\to+\infty} \frac{\cancel{1}+\frac{1}{2}\cdot\frac{1}{x+1}+o\left(\frac{1}{x}\right)\cancel{-1}+\frac{1}{2}\left(\frac{+1}{x+2}\right)+o\left(\frac{1}{x}\right)}{\frac{1}{x}} =$$

$$= \lim_{x\to+\infty}\left(\frac{\frac{1}{x+1}+\frac{1}{x+2}}{2\cdot\frac{1}{x}}\right) = \lim_{x\to+\infty}\left(\frac{x}{2(x+1)}+\frac{x}{2(x+2)}\right) = \frac{1}{2}+\frac{1}{2}=1$$

$$\frac{1}{2\left(1+\frac{1}{x}\right)} \to \frac{1}{2\cdot(1+0)} = \frac{1}{2}$$

$\frac{1}{2}$

13

$b_n \to 0$

$$\lim_{n\to+\infty} \sqrt{n} \cdot \sin\left(\sqrt{n+1}-\sqrt{n}\right) = \qquad a_n \frac{\sin(b_n)}{b_n} \cdot b_n$$

$$\frac{\sqrt{n+1}-\sqrt{n}}{1} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \qquad \sin b_n \approx b_n$$

$$= \lim_{n\to+\infty} \left(\underbrace{\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}}_{\frac{1}{\sqrt{1+\frac{1}{n}}+1} \to \frac{1}{2}} \cdot \underbrace{\boxed{\frac{\sin\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right)}{\frac{1}{\sqrt{n+1}+\sqrt{n}}}}}_{\to 1} \right) = \frac{1}{2}$$

$$\frac{\sin x}{x} \to 1$$