Esercitazioni di Analisi 1

Exe 11

18 Novembre 2022

PROBLEMI PROPOSTI

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1) $\displaystyle\lim_{x\to 0} \frac{\cos(\ln(\cos x)) - 1}{x^\alpha}$

2) $\displaystyle\lim_{x\to \frac{\pi}{2}} \frac{\ln(\sin x)}{\cos x}$

3) $\displaystyle\lim_{x\to +\infty} \frac{\left(\frac{\pi}{2} - \arctan x\right)\ln(1+x)}{\ln(1+x^\alpha)}$

4) $\displaystyle\lim_{n\to +\infty} \frac{\tan\left(\frac{1}{n!}\right) - \sin\left(\frac{1}{n!}\right)}{1-\cos\left(\frac{1}{n^n}\right)}$

5) $\displaystyle\lim_{x\to 0} \frac{\sin x + \sin 2x + \dots + \sin 100x}{\sin(\sin(\sin(x)))}$

6) $\displaystyle\lim_{x\to 0} \frac{2\sin x - \sin 2x}{x^\alpha}$

NUOVI

7) DATI $A, B \subset \mathbb{R}$ TALI CHE $d(A,B) = 0$, È VERO CHE $A \cap B \neq \emptyset$?

8) COME (7) MA CON A, B CHIUSI

9) COME (7) MA CON A, B COMPATTI

13) (AGGIUNTO DURANTE LA LEZIONE) PER OGNI $n \in \mathbb{N}$ SIA $A_n \subset \mathbb{R}$ E SIA $A_{n+1} \subset A_n$. È VERO CHE $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$? E SE AGGIUNGO L'IPOTESI CHE SIANO CHIUSI? E CHE SIANO COMPATTI?

10) MOSTRARE CHE SE $A \subset \mathbb{R}$ $f: \mathbb{R} \to \mathbb{R}$ t.c. $f(x) = d(x, A)$ È CONTINUA.

11) $\displaystyle\lim_{x\to +\infty} \left(\frac{\alpha + \sin x}{10 + \cos x}\right)^x$

12) SE $f: \mathbb{R} \to \mathbb{R}$ PERIODICA CON $T = 0$ E CONTINUA ALLORA È COSTANTE.

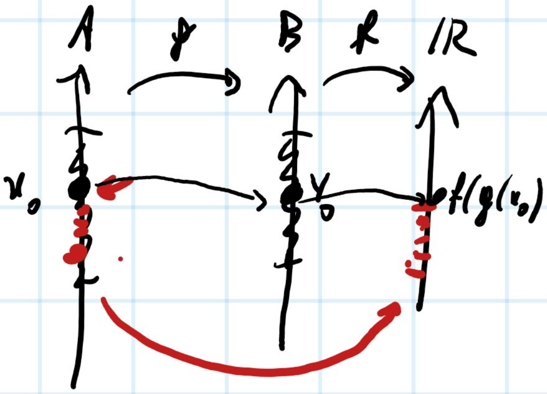
T. DATI $A, B \subset \mathbb{R}$ $\quad g: A \to B \quad f: B \to \mathbb{R}$

g CONTINUA IN $x_0 \in A$, f CONTINUA IN $\underbrace{y_0 = g(x_0)}$.

ALLORA $f(g(x))$ È CONTINUA IN x_0
(?)

DIM

$\boxed{\text{DEVO DIM. CHE } \forall (a_n) \text{ A VALORI IN } A \text{ T.C. } a_n \to x_0 \text{ SI HA CHE } f(g(a_n)) \to f(g(x_0)) \text{ ?}}$



Prendo (a_n) in A t.c. $\boxed{a_n \to x_0}$

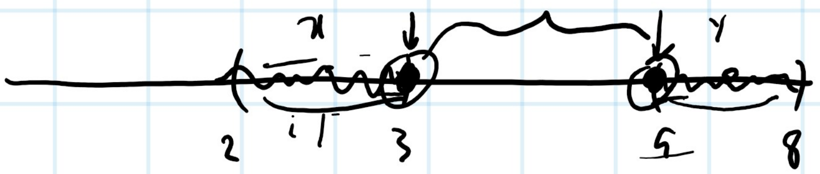
POICHÉ g È CONT. IN x_0 SI HA

$g(a_n) \to g(x_0)$

MA $(g(a_n))$ È SUCCESS. A VALORI IN B T.C. $\underbrace{g(a_n)} \to \underbrace{g(x_0)} = y_0$

$$\boxed{f(g(a_n)) \to f(g(x_0))}$$

7

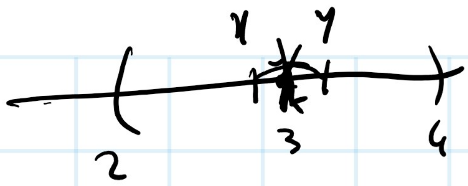


DEF. DATI $A, B \subset \mathbb{R}$ DEFINIAMO

$$d(A,B) = \inf\{d(x,y) \mid x \in A, y \in B\}$$

$\updownarrow$

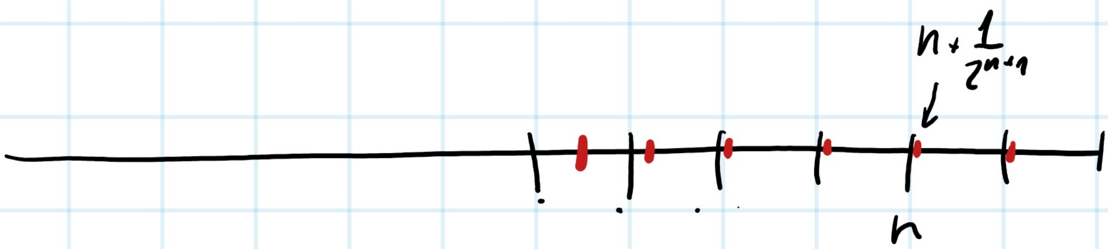
$|x-y|$



$A =]2,3[$

$B =]3,4[$

8



$A = \mathbb{N}$

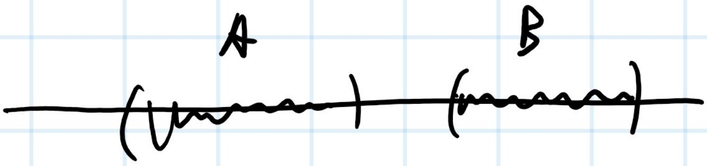
$B = \{ n + \frac{1}{2^{n+1}} \mid n \in \mathbb{N} \}$

$d\left(n, n + \frac{1}{2^{n+1}}\right) = \frac{1}{2^{n+1}} \to 0$

(9) A, B

$\{d(x,y) \mid x\in A,\ y\in B\}$

inf



$\forall \varepsilon>0\ \exists\ x\in A,\ y\in B$ t.c. $|x-y|<\varepsilon$

$\varepsilon=1$ $\quad x_1\in A \quad y_1\in B$ t.c. $|x_1-y_1|<1$

$\vdots$

$\varepsilon=\frac{1}{n}$ $\quad x_n\in A \quad y_n\in B$ t.c. $|x_n-y_n|<\frac{1}{n}$

$\vdots$

(x_n) È IN A $\qquad$ (y_n) È IN B

$\exists\ (x_{n_k})$ T.C. $x_{n_k}\to x_0\in A$

Prendo $\boxed{(y_{n_k})}$ LA CORR. SOTTOSUCC. DI (y_n)

E Prendo $(y_{n_{k_p}})$ SS. DI $\boxed{(y_{n_k})}$ T.C. $(y_{n_{k_p}})\to y_0$

$\boxed{d(x_0,y_0)=d(A,B)}$?

$(x_{n_{k_p}})\to x_0$

$$d(x_0, y_0) \geq d(A,B)$$

$$d(x_0, y_0) \leq d(A,B)$$

$\forall \varepsilon > 0$ $\quad \boxed{d(x_0, y_0)} \leq \overbrace{d(A,B) + \varepsilon}$

$\uparrow$

0

$A \quad B$

$\Downarrow \quad \Downarrow$

$x_0 = y_0$

$$|x_0 - y_0| = |x_0 - x_{n_{k_p}} + x_{n_{k_p}} - y_{n_{k_p}} + y_{n_{k_p}} - y_0| \leq$$

$$\leq |x_0 - x_{n_{k_p}}| + |x_{n_{k_p}} - y_{n_{k_p}}| + |y_{n_{k_p}} - y_0| \leq$$

$\downarrow \quad \downarrow \quad \downarrow$

$0 \quad 0 \quad 0$

$\leq \frac{1}{n_{k_p}} \to 0$

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

$x_{n_{k_p}} \to x_0$

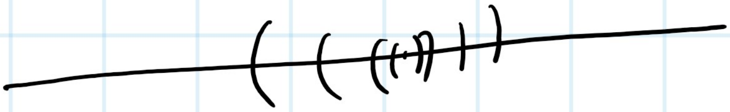
$y_{n_{k_p}} \to y_0$

13

$\{A_n\}_n$ $\qquad$ $A_1 \supset A_2 \supset A_3 \supset \cdots \supset A_n \supset$

$A_n \neq \emptyset$ $\qquad$ $A_n \supset A_{n+1}$

$\forall n$



$\bigcap A_n \neq \emptyset$

A_n CHIUSI $\qquad$ $[n, +\infty)$

A_n COMPATTI

$A_n = [n, +\infty)$

$\bigcap A_n = \emptyset$

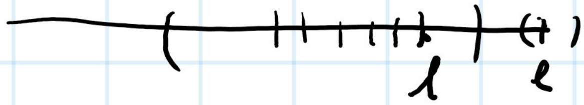
A_n compatti

$\forall n \in \mathbb{N}$ prendo $x_n \in A_n$

$(x_{n_k}) \to x_0 \in A_1$

$\forall n \in \mathbb{N}$ $\quad x_{n_k} \in A_n$ DEF. in k

$x_{n_k} \to x_0 \quad x_0 \in A_n$



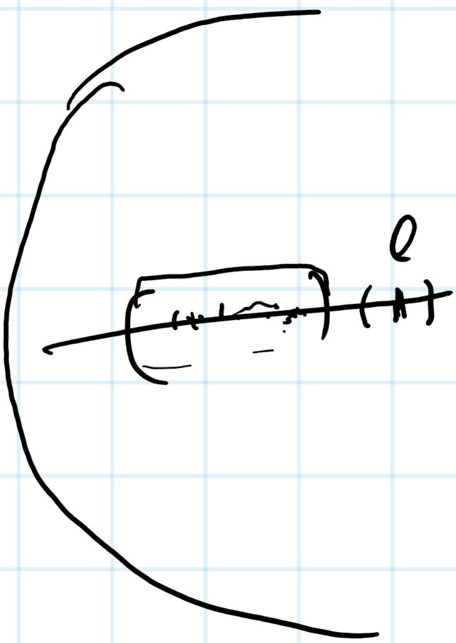
$A_1 \supset \cdots \supset A_n \supset A_{n+1} \supset \cdots$

A_n COMPATTI $\forall n$ NON VUOTI

$\forall n \in \mathbb{N} \quad x_n \in A_n$

$(x_n) \subset A_1 \qquad x_{n_k} \to x_0 \in A_1$

$$\forall n \in \mathbb{N} \quad \left.\begin{array}{l} (x_{n_k}) \overset{\text{DEF.}}{\subset} A_n \\ x_{n_k} \to x_0 \end{array}\right\} \Rightarrow x_0 \in A_n$$



$x_0 \in \bigcap A_n$

3

$$\lim_{x\to+\infty} \frac{\left(\frac{\pi}{2} - \arctan x\right)\cdot \ln(1+x^x)}{\ln(1+x)} \approx \lim_{x\to+\infty} \frac{\frac{1}{x}\cdot \cancel{x}\cancel{\ln x}}{\cancel{\ln x}} = 1$$

$$\ln(1+x^x) \approx \ln x^x = x\ln x$$

(?)

$$f(h(x) + \cancel{o(h(x))}) \approx f(h(x))$$

$$\ln(1+x^x) = \ln\left(x^x\cdot\left(1+\frac{1}{x^x}\right)\right) =$$

$$= \ln x^x + \ln\left(1+\frac{1}{x^x}\right) \approx \ln(x^x) =$$

$$h(x) = o(k(x))$$

$$\sqrt{h(x)+k(x)} = \sqrt{k(x)}\cdot\sqrt{\frac{h(x)}{k(x)}+1}$$

$$\frac{h(x)}{k(x)} \to 0$$

$x>0$

$$\arctan x + \arctan\frac{1}{x} = \frac{\pi}{2}$$

$$\boxed{\frac{\pi}{2} - \arctan x} = \boxed{\arctan\frac{1}{x}} \approx \boxed{\frac{1}{x}}$$

$$\lim_{y\to0}\frac{\arctan y}{y} = 1$$

$$\arctan y \approx y$$

6) $\lim_{x\to0^+} \dfrac{2\sin x - \sin 2x}{x^\alpha}$

$\alpha = 1$

$$\left(\underbrace{2\,\frac{\sin x}{x}}_{2\cdot 1} - \underbrace{\frac{\sin 2x}{2x}\cdot 2}_{1\cdot 2}\right) \quad 2-2=0$$

$\alpha = 2$

$$\lim_{x\to0^+} \frac{2\sin x - 2\sin x\cos x}{x^\alpha} = \lim_{x\to0^+} \frac{\overbrace{2\sin x(1-\cos x)}^{2x\cdot\frac{1}{2}x^2}}{x^\alpha} =$$

$$= \begin{cases} +\infty & \alpha > 3 \\ 1 & \alpha = 3 \\ 0 & \alpha < 3 \end{cases}$$

$$\boxed{\lim_{x\to0^+} \frac{\tan x - \sin x}{x^3} = \frac{1}{2}}$$

$$\lim_{x\to0^+} \frac{\tan x - x}{x^2} =$$

$\boxed{\sin x < x < \tan x}$

$$\frac{\tan x}{x^2} - \frac{x}{x^2}$$

$$(+\infty) - (+\infty)$$

$\lim \frac{\tan x}{x} = 1$

$\tan x = x + \underbrace{o(x)}$

$\tan x = x + o(x^\alpha)$

$\alpha < 3$

$$\frac{0}{x^2} < \boxed{\frac{\tan x - x}{x^2}} < \boxed{\frac{\overbrace{\tan x - \sin x}^{\sim \frac{1}{2}x^3}}{x^2}}$$

$$\downarrow \frac{2}{2}\quad\downarrow 0$$

$\underbrace{\tan x - x} = o(x^\alpha) \quad \alpha < 3$

$$\boxed{\tan x - x = O(x^3)}$$

$$\underset{\downarrow\ 0}{0} < \frac{\tan x - x}{x^3} < \underset{\downarrow\ \frac{1}{2}}{\frac{\tan x - \sin x}{x^3}}$$

11

$$\lim_{x \to +\infty} \left(\frac{8 + \sin x}{10 + \cos x} \right)^x = 0$$

$\lambda < 1$

$= \max_{\mathbb{R}}$

$$\left(1 - \frac{1}{x}\right)^x \not\to 0$$

$1 - \frac{1}{x}$

$$(\;)^x \leq (0{,}99)^x$$

$\downarrow$ 0

$$0 \leq (\;)^x \leq \lambda^x$$

$\downarrow$ 0 $\quad \downarrow$ 0 $\quad \downarrow$ 0