Esercitazioni di Analisi 1

Exe 12

2 Dicembre 2022

PROBLEMI PROPOSTI

1) DATA $f:\mathbb{R}\to\mathbb{R}$ CONTINUA ED $A\subset\mathbb{R}$, DIRE SE SONO VERE O FALSE:

a) A CHIUSO $\Rightarrow$ $f(A)$ CHIUSO ?

b) A APERTO $\Rightarrow$ $f(A)$ APERTO ? (E $f^{-1}(A)$?)

c) A INTERVALLO $\Rightarrow$ $f(A)$ INTERVALLO ? (E $f^{-1}(A)$?)

2) DATA $f:\mathbb{R}\to\mathbb{R}$ DIRE SE SONO VERE O FALSE:

a) f MANDA COMPATTI IN COMPATTI $\Rightarrow$ f CONTINUA ?

b) .... INTERVALLI ... INTERVALLI .... ?

3) DATA $f:\mathbb{R}\to\mathbb{R}$ DIRE SE f HA UN PUNTO FISSO NEI SEGUENTI CASI:

$\to$ **a)** f CONTINUA E LIMITATA

b) f CONTINUA E DECRESCENTE $\leftarrow$ SI

c) f CONTINUA E CRESCENTE $\leftarrow$ NO

$\Rightarrow$ **d)** f LIPSCHITZIANA CON COSTANTE $L=\frac{1}{2}$ $\leftarrow$

4) DATA $f:A\to A$ CONTINUA, DIRE SE f HA UN PUNTO FISSO NEI SEGUENTI CASI:

a) $A=(0,1)$

b) $A=[0,1]$ $\leftarrow$

c) $A=(0,1]$

d) $A=\mathbb{R}-\{0\}$

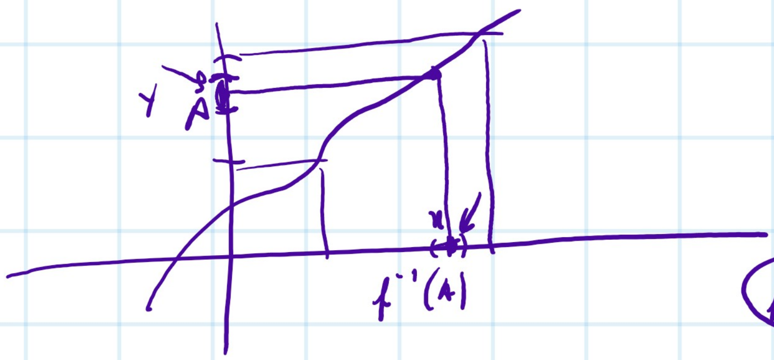
1a Falso

$f(x) = \arctan x \qquad A = [0, +\infty)$

$$f(A) = [0, \tfrac{\pi}{2})$$

1b.1 $f(x) = k =$ costante $\qquad A = (0,1) \qquad f(a) = x$

$f: \mathbb{R} \to \mathbb{R}$ CONTINUA $\quad A \subset \mathbb{R}$ APERTO $\overset{\text{Sì}}{\Longrightarrow} f^{-1}(A)$ Aperto

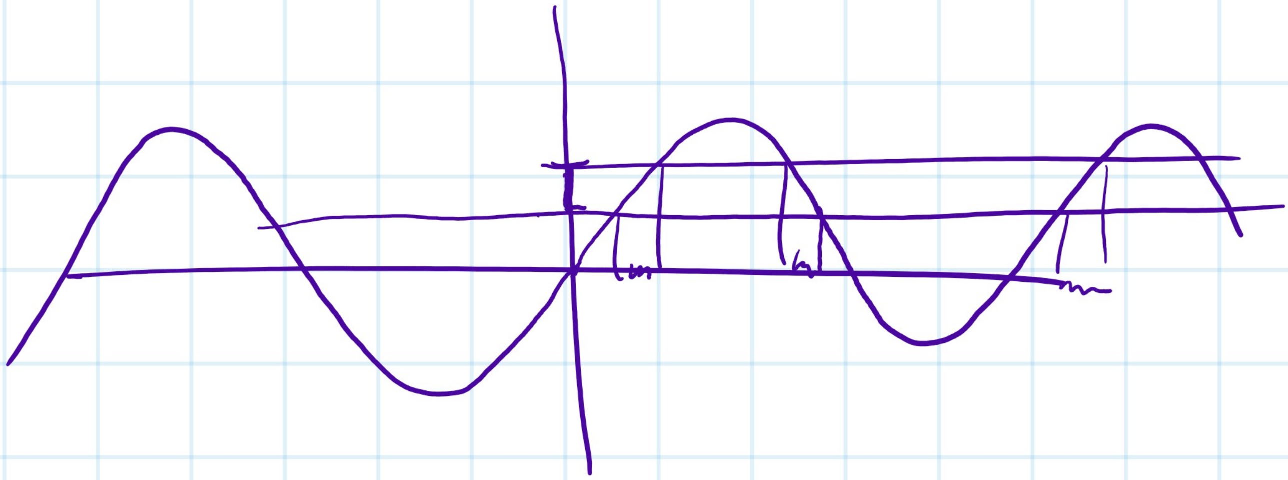


f è continua in x

$x \in f^{-1}(A) \Rightarrow \exists\, y \in A$ t.c. $f(x) = y \Rightarrow \exists\, I$ intorno di y t.c. $I \subset A \Rightarrow$

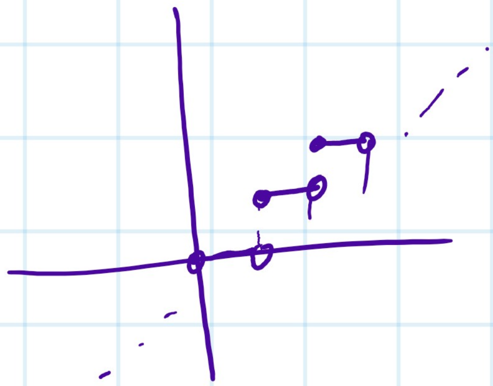
$\Rightarrow \exists\, J$ intorno di x t.c. $f(J) \subset I \Rightarrow J \subset f^{-1}(A) \Rightarrow$ x È INTERNO AD A

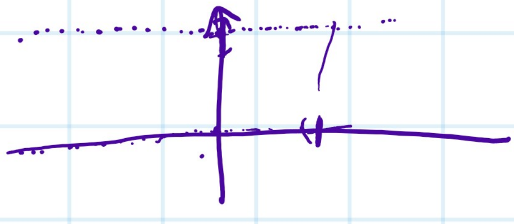
$I \subset A$



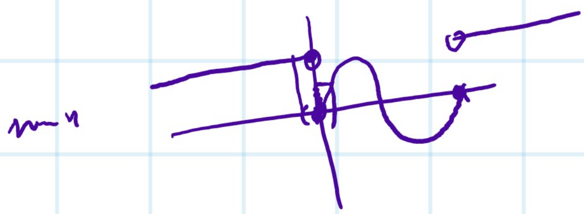
2a **FALSO**

$f(x) = [x]$





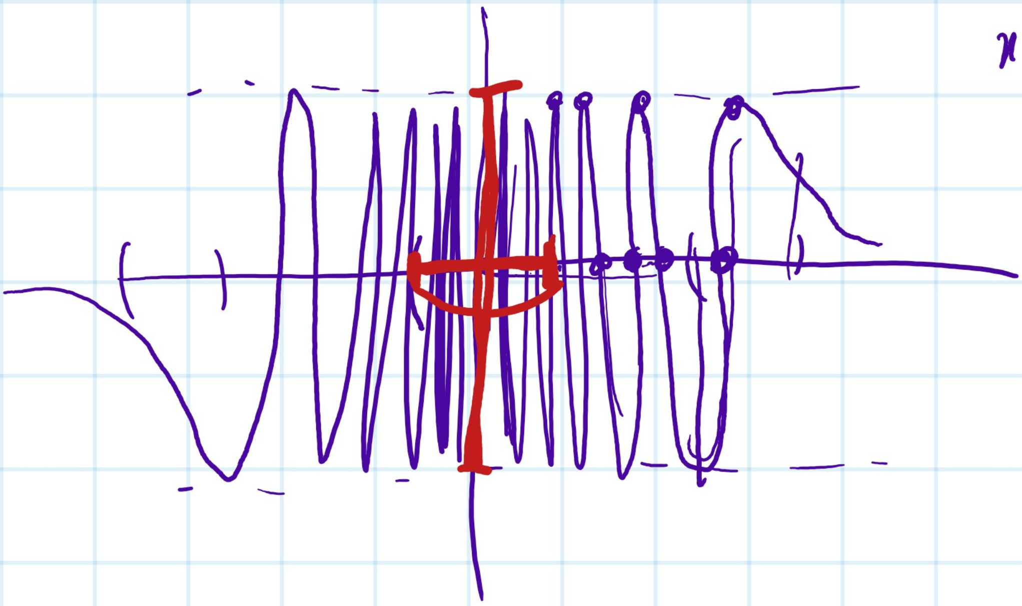
2b



$$f(x) = \begin{cases} 0 & x = 0 \\ \sin\frac{1}{x} & x \neq 0 \end{cases}$$

$$\frac{1}{x} = k\pi$$

$$x = \frac{1}{k\pi}$$



$$a_n = \frac{1}{n\pi} \qquad f(a_n) = 0 \to$$

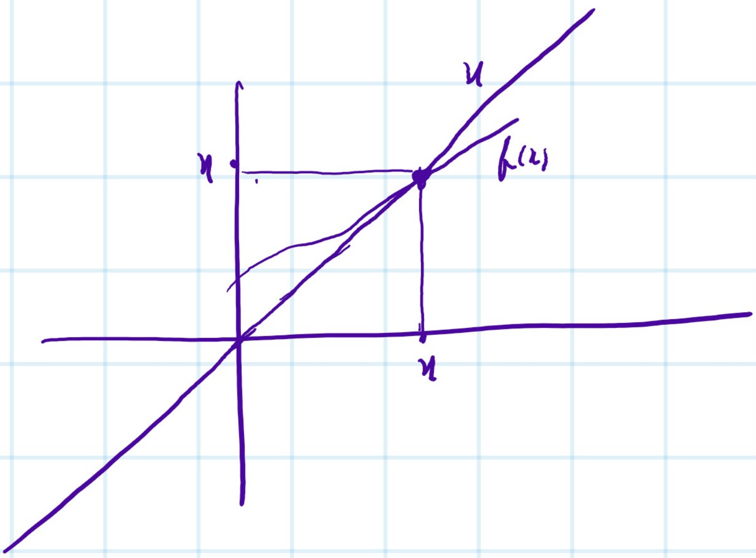
$$b_n = \frac{1}{\frac{\pi}{2} + 2n\pi} \qquad f(b_n) = 1 \to 1$$

$f: A \to A$ HA PUNTO FISSO

$\exists x \in A$ t.c. $f(x) = x$

$\boxed{3a}$ $\boxed{SI}$

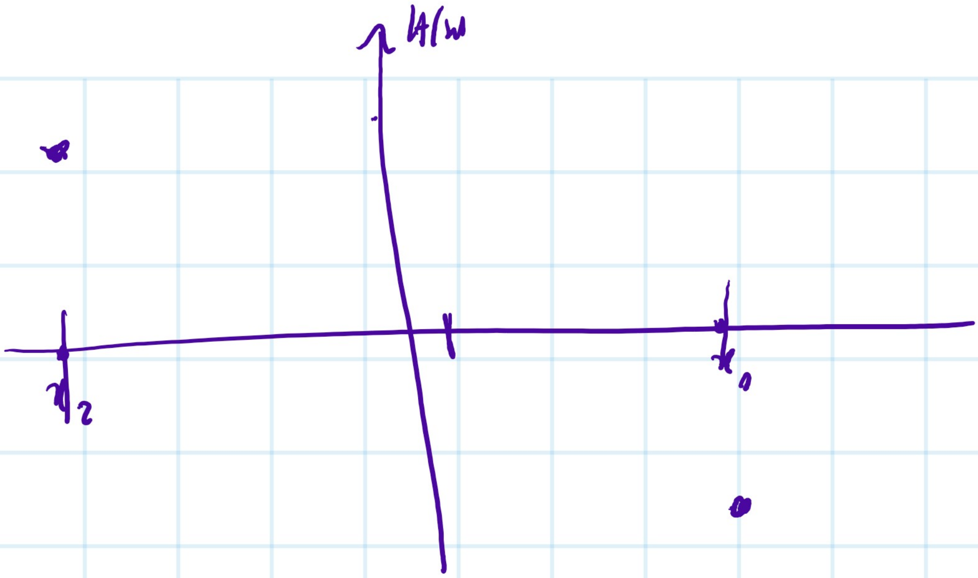
$f: \mathbb{R} \to \mathbb{R}$ CONT. E LIMITATA $\Rightarrow$ $\exists x \in \mathbb{R}$ t.c. $f(x) = x$



$$\boxed{H(x)} = f(x) - x = 0$$

$\exists x_1 > 0$ t.c. $H(x_1) < 0$ $\Leftarrow$ $\lim_{x \to +\infty} \boxed{H(x)} = \lim_{x \to +\infty} \left(f(x) - x \right) = -\infty$

$\exists x_2 < 0$ t.c. $H(x_2) > 0$ $\Leftarrow$ $\lim_{x \to -\infty} H(x) = \lim_{x \to +\infty} f(x) - x = +\infty$



T.V.I. $\Rightarrow$ $\exists\, x \in (x_1, x_2)$ t.c. $H(x) = 0$

$f(x) - x = 0$

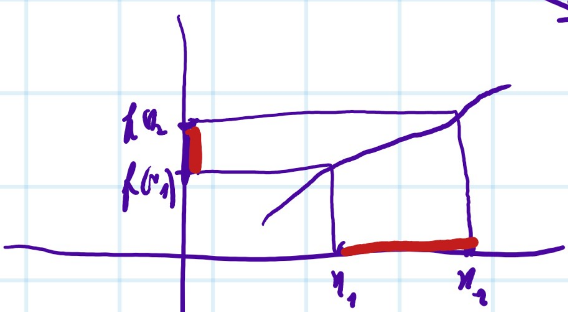
$\boxed{f(x) = x}$

$\boxed{31}$ **VERO**

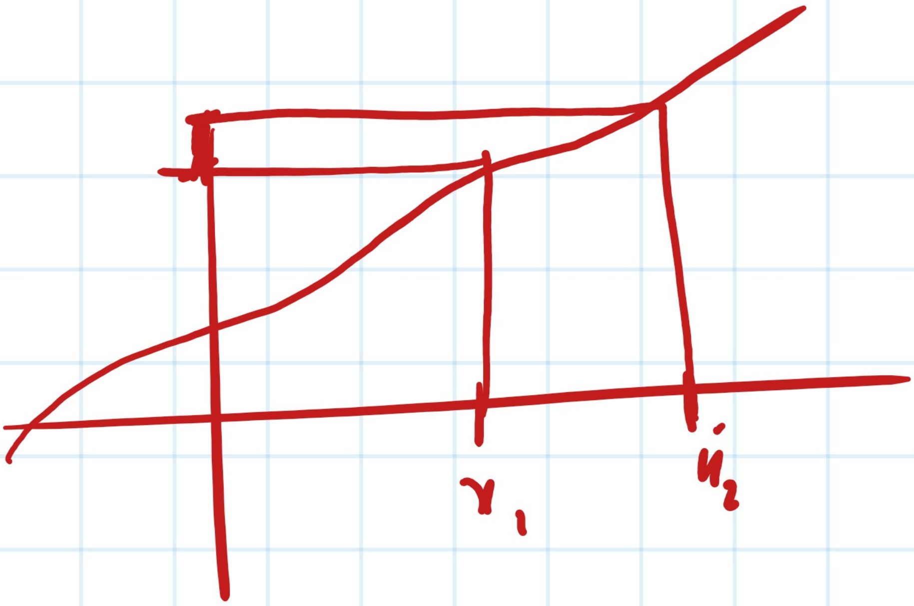
$f: \mathbb{R} \to \mathbb{R}$ È $\underbrace{\text{LIP. CON COST. } \tfrac{1}{2}}$ $\Rightarrow$ $\exists!\, x \in \mathbb{R}$ t.c. $f(x) = x$

$\forall x_1, x_2 \in \mathbb{R}$

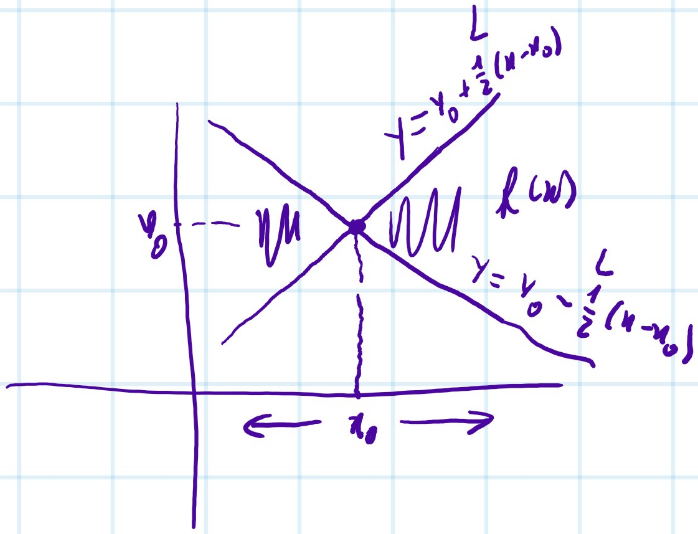
$|f(x_1) - f(x_2)| \leq \frac{1}{2} |x_1 - x_2|$



$$|f(x_1) - f(x_2)| \leq \frac{1}{2} |x_1 - x_2|$$



$H(x) = f(x) - x = 0$



$\boxed{\text{LEMMA}}$ DATA $f: \mathbb{R} \to \mathbb{R}$ LIP. CON COST. L

ALLORA $\forall x \geq x_0$

(1) $f(x_0) - L(x - x_0) \leq f(x) \leq f(x_0) + L(x - x_0)$

$\forall x \leq x_0$

(2) $f(x_0) + L(x - x_0) \leq f(x) \leq f(x_0) - L(x - x_0)$

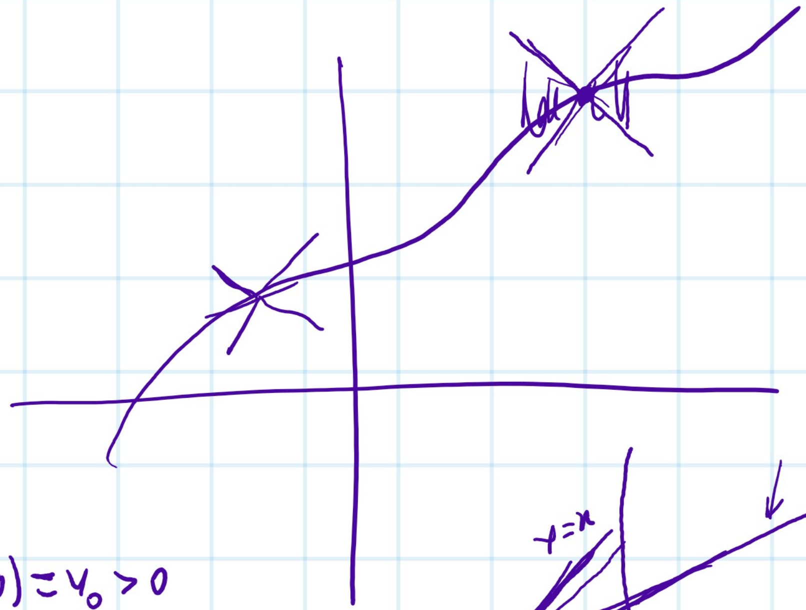
$\boxed{\forall x \geq x_0}$

$\boxed{\text{DIM}}$

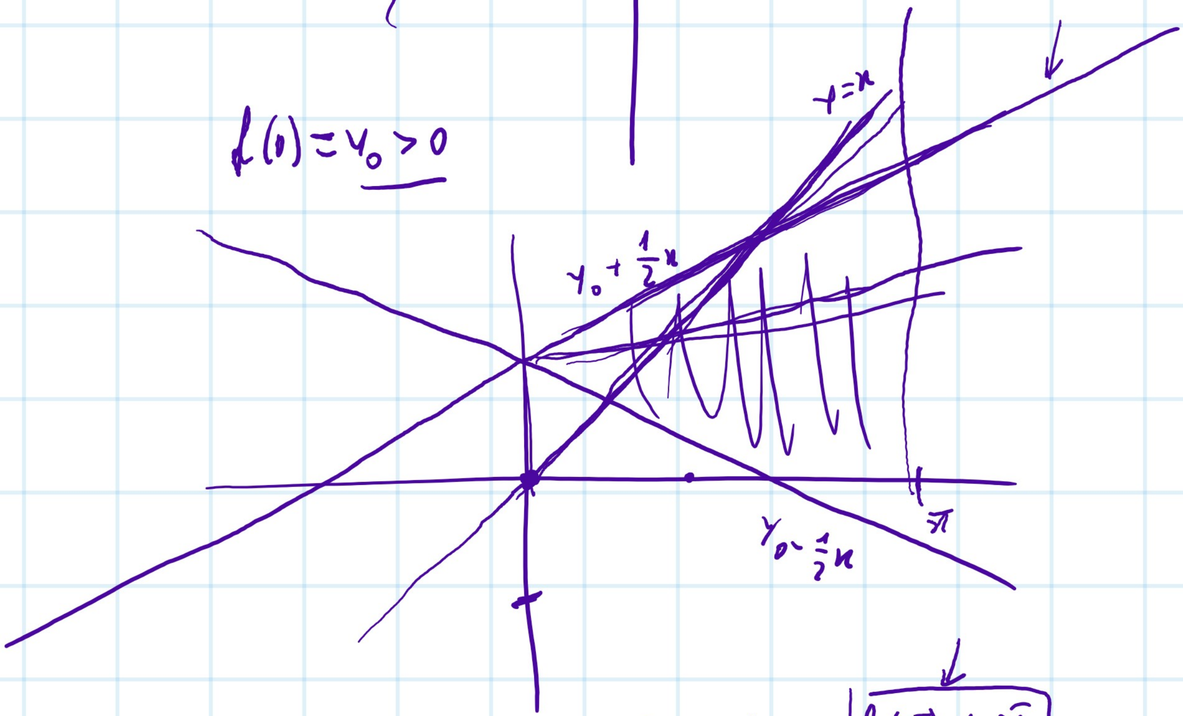
$$|f(x)-f(x_0)| \leq L\,|x-x_0|$$

$$-L(x-x_0) \leq f(x)-f(x_0) \leq L(x-x_0)$$

$$\underline{f(x_0)-L(x-x_0)} \leq f(x) \leq \underline{f(x_0)+L(x-x_0)}$$



$f(0)=\underline{y_0>0}$



$f(0)=y_0>0$ $\quad\exists \bar{x}>0$ t.c. $\boxed{f(\bar{x})<\bar{x}}$

$f(x) < \boxed{y_0+\frac{x}{2}<x}$

$H(x)=f(x)-x$ $\quad H(0)=y_0-0>0$

$H(\bar{x})=f(\bar{x})-\bar{x}<0$

$y_0<\frac{x}{2}$

$H(x)$ $\qquad \underbrace{[\overset{\downarrow}{0}, \overset{\downarrow}{\bar{x}}]}$ $\qquad \exists\, \tilde{x} \in [0, \bar{x}]$ t.c. $H(\tilde{x}) = 0$

$x_1 , x_2 \qquad x_1 \neq x_2 \qquad f(x_1) = x_1$

$f(x_2) = x_2$

$$|f(x_1) - f(x_2)| \leq \frac{1}{2} |x_1 - x_2|$$

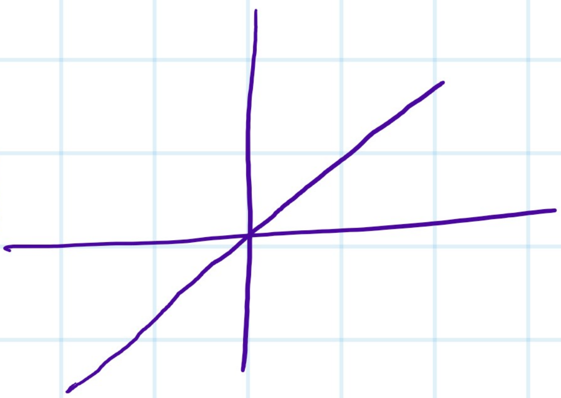
$$||$$

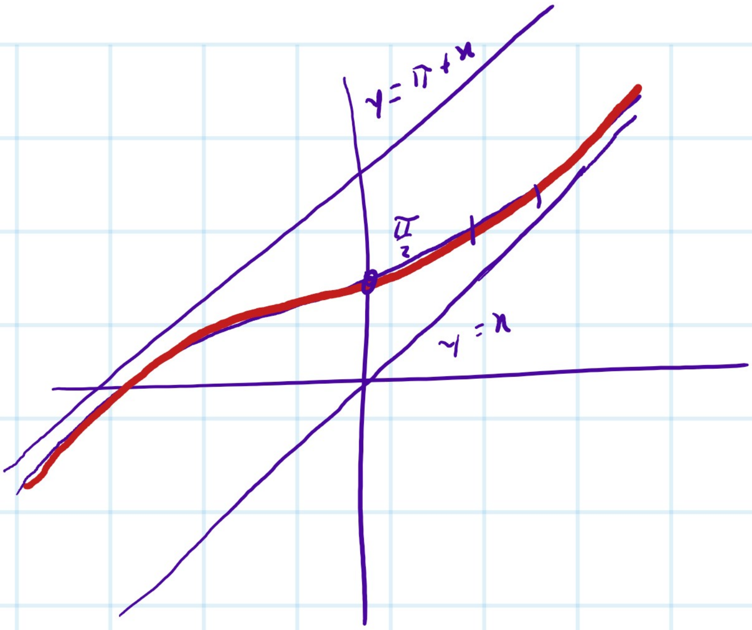
$$|x_1 - x_2|$$

$$f : \underbrace{(0, +\infty)} \to \underbrace{(0, +\infty)}$$

$$\forall x, y \quad |f(x) - f(y)| < |x - y|$$

$$\boxed{f(x) = x + \frac{\pi}{2} - \arctan x}$$





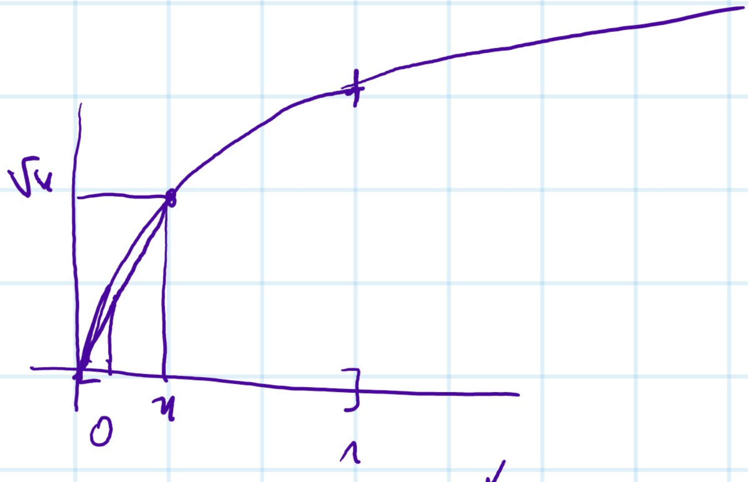
$\leq 0{,}99\,|x-y|$

$< |x-y|$

$|f(x) - f(y)| < |x-y|$

$f(x) = \sqrt{x}$ su $[0,1]$ U.C.?

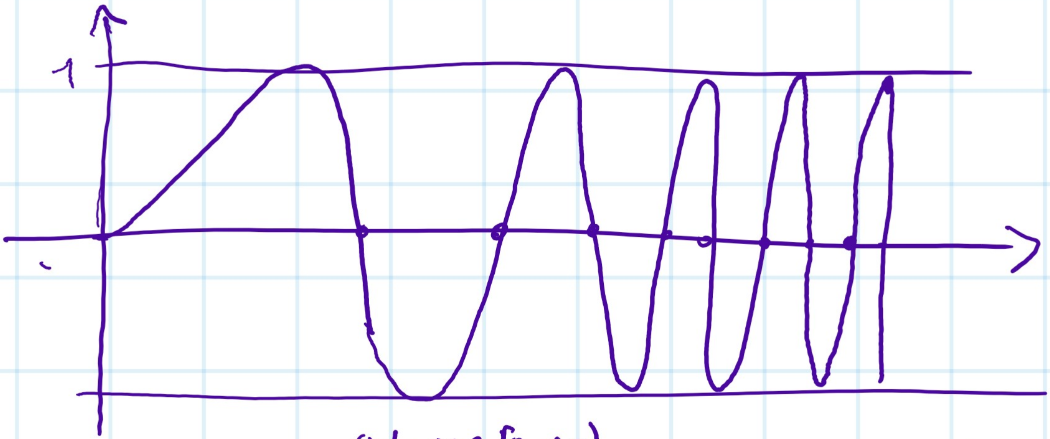
$[0, +\infty)$



$|f(x) - f(0)| = \sqrt{x} \leq L\,x$

U.C.

$f(x) = \sin(x^2)$ $[0,1]$ $\boxed{[0,+\infty)}$



$\rightarrow$ " $\forall \varepsilon > 0 \ \exists \delta > 0$ t.c. $\underbrace{(\forall x, y \in [0,+\infty))}$ $|x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon$ "

$$\boxed{\exists \varepsilon_0 > 0 \text{ t.c. } \forall \delta > 0 \ \exists x_1, x_2 \in [0,+\infty) \text{ t.c. } |x_1 - x_2| < \delta, \ |f(x_1) - f(x_2)| > \varepsilon_0}$$

$\frac{1}{2}$

$a_n = \sqrt{\frac{\pi}{2} + 2n\pi}$

$b_n = \sqrt{2n\pi}$

$\boxed{|a_n - b_n|} = \sqrt{\frac{\pi}{2} + 2n\pi} - \sqrt{2n\pi} =$

$= \dfrac{\frac{\pi}{2}}{\sqrt{\frac{\pi}{2} + 2n\pi} + \sqrt{2n\pi}} \rightarrow 0$

$f(x) = \sin x^2$

$\boxed{f(a_n) - f(b_n)} = \sin\left(\frac{\pi}{2} + 2n\pi\right) - \sin(2\pi n) =$

$= 1 - 0 = 1$

$(0,+\infty)$

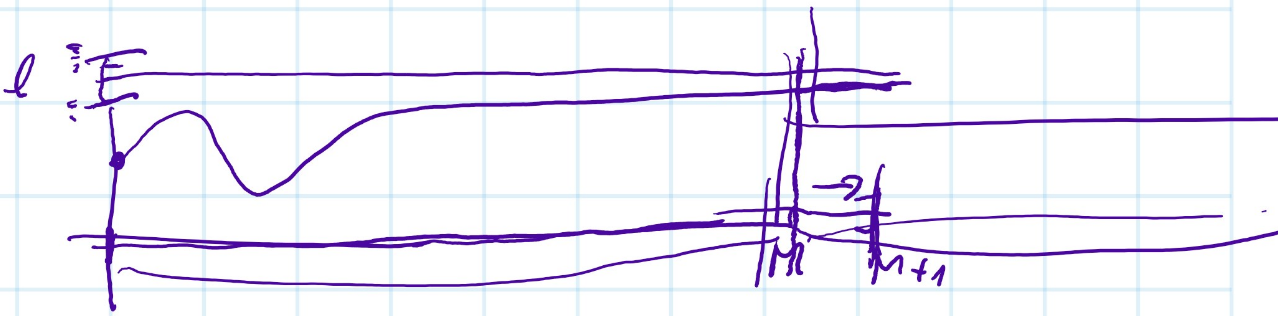
$f(x) = x \sin\frac{1}{x}$

$$F(x) = \begin{cases} 0 & x=0 \\ f(x) & x\in(0,1] \end{cases}$$

$(0,1]$ $[0,1]$

K

$[1,+\infty)$



$\forall \varepsilon>0\ \exists \delta>0$ t.c. . . - - - .