Esercitazioni di Analisi 1

Exe 13

5 Dicembre 2022

PROBLEMI PROPOSTI

DIRE SE LE FUNZIONI SEGUENTI SONO UNIFORMEMENTE CONTINUE E/O LIPSCHITZIANE NEGLI INSIEMI INDICATI A FIANCO:

1) $f(x) = \sqrt[3]{x + x^3}$ SU $[-1,1]$, $[1,+\infty)$, $\mathbb{R}$.

2) $f(x) = \frac{\sin x}{x}$ SU $(0,1)$, $(0,+\infty)$.

3) $f(x) = \sqrt[3]{x + x^5}$ SU $[-1,1]$, $[1,+\infty)$, $\mathbb{R}$.

4) $f(x) = \sqrt{2+\sin x}$ SU $\mathbb{R}$.

5) $f(x) = \left(\arctan \frac{1}{x}\right) \cdot \sin x$ SU $(0,+\infty)$ [VARIANTI: $\boxed{(\arctan x)(\sin \frac{1}{x})}$, $\boxed{(\arctan x)(\sin x)}$, $\boxed{(\arctan \frac{1}{x})(\sin \frac{1}{x})}$

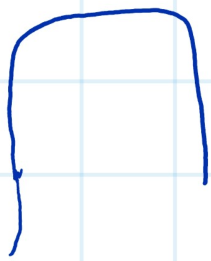
6) $f(x) = x \sin x$ SU $[-100, 100]$, $\mathbb{R}$.

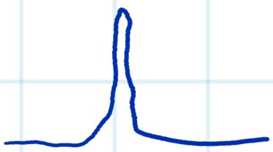
7) $f(x) = |\sin x|^x$ SU $[1, 100]$, $\mathbb{R}$.

8) $f(x) = |\sin x|^{\frac{1}{x}}$ SU $[1, 100]$, $\mathbb{R}$.

9) $f(x) = d(x,\mathbb{Z})$ SU $\mathbb{R}$.

10) SIA $f:\mathbb{R}\to\mathbb{R}$ TALE CHE $f(\ln n)=0$ $\forall n \in \mathbb{N}-\{0\}$. DIRE SE $\lim_{x\to+\infty} f(x)=0$ NEL CASO CHE f SIA: (a) CONTINUA, (b) UNIFORMEMENTE CONTINUA, (c) LIPSCHITZIANA.
QUANDO È POSSIBILE, STIMARE L'ORDINE DI INFINITESIMO.





SOLUZIONI

① $\boxed{f(x) = \sqrt[3]{x + x^3}}$ $\qquad [-1, 1] \qquad [1, +\infty)$

$\mathbb{R}$

SU $[-1,1]$ CONT. QUINDI U.C. PER H.C.

$$f(x) = \underbrace{\left(\sqrt[3]{x + x^3} - x\right)}_{g(x)} + \underbrace{x}$$

$\boxed{\text{T.1}}$ DATE $f, g : A \to \mathbb{R}$ U.C. SU A ALLORA

$f + g$ È UNIF. CONT. SU A.

$\boxed{\text{DIM}}$ (?) $\boxed{\forall \varepsilon > 0 \ \exists \delta > 0 \text{ t.c. } \forall x, y \in A \ |x - y| < \delta \Rightarrow}$

$\Rightarrow \left| \left(f(x) + g(x)\right) - \left(f(y) + g(y)\right) \right| < \varepsilon$

(S1) $\Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}$

$|g(x) - g(y)| < \frac{\varepsilon}{2}$

$\leq |f(x) - f(y)| + |g(x) - g(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$$g(x) = \boxed{\sqrt[3]{x+x^3} - x} = \qquad [0, +\infty)$$

$$= x\left(\sqrt[3]{1+\frac{1}{x^2}} - 1\right) \overset{\text{per } x\to+\infty}{\approx} x \cdot \left(\frac{1}{3}\cdot\frac{1}{x^2}\right) = \frac{1}{3}\cdot\frac{1}{x} \to 0$$

$$(1+f(x))^\alpha - 1 \approx \alpha f(x)$$

$$=$$

T.2 $f: [a,+\infty) \to \mathbb{R}$ CONT. E TALE CHE $\lim\limits_{x\to+\infty} f(x) = 0$ (1)

ALLORA f È U.C. SU $[a,+\infty)$

DIM

(?) $\forall \varepsilon>0 \ \exists \delta>0 \ \forall x,y \in [a,+\infty) \quad |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$

DA (1) SEGUE CHE $\exists M>a$ t.c. $\forall x \in [M,+\infty)$ SI HA $|f(x)| < \frac{\varepsilon}{4}$

SU $[a,M]$ PER H.P. $\exists \delta>0$ t.c. $\forall x,y \in [a,M]$ $|x-y|<\delta \Rightarrow$

$\Rightarrow |f(x)-f(y)| < \frac{\varepsilon}{2}$

$\forall x,y \in [a,+\infty)$ SE $x,y \in [M,+\infty)$ ALLORA $|f(x)-f(y)| \leq$

$\leq |f(x)|+|f(y)| < \frac{\varepsilon}{4}+\frac{\varepsilon}{4} = \frac{\varepsilon}{2} < \varepsilon$

SE $\underbrace{x<M<y}$ $\quad |x-y|<\delta \quad |x-M|<\delta$

$$|f(x) - f(y)| = |f(x) - f(M) + f(M) - f(y)| \leq$$

$$\leq \underbrace{|f(x) - f(M)|} + \underbrace{|f(M) - f(y)|} \leq$$

$$\leq |f(x) - f(M)| + |f(M)| + |f(y)| \leq$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon$$

SE $x, y \leq M$ $\quad x, y \in [a, M]$ $\quad |x - y| < \delta$ $\quad |f(x) - f(y)| < \frac{\varepsilon}{2}$

$< \varepsilon$

ESEMPIO CATTIVO

$A \cap B = \emptyset$

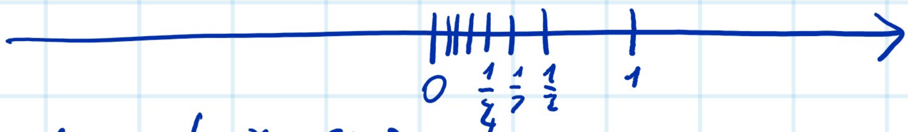
$f : A \cup B \to \mathbb{R}$

f U.C. SU A

f U.C. SU B $\quad$ MA f NON U.C. SU $A \cup B$

$$A = \underbrace{\left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N} - \{0\} \right\}} \cup \{0\}$$

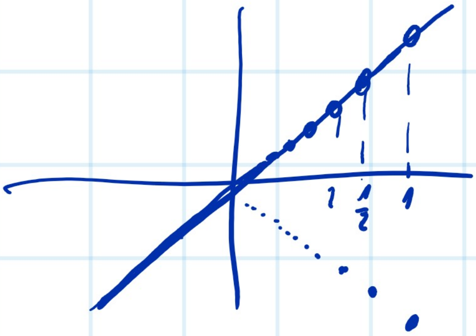
$$B = \mathbb{R} - \boxed{m}$$



$$f(x) = \begin{cases} x & \text{su } B \\ -x & \text{in } A - \{0\} \end{cases}$$

$f\big|_B$ È LIP. (E U.C.)

$f\big|_A$ È LIP. (U.C)



$\boxed{\text{T.3}}$ DATI $A=(a,c]$ E $B=[c,b)$ $\quad a,b\in\mathbb{R}^*$

$\boxed{\text{INCOLLAMENTO}}$ SE $f:(a,b)\to\mathbb{R}$ E U.C. SU $(a,c]$ E SU $[c,b)$

ALLORA È U.C. SU (a,b)

$\boxed{\text{DIM.}}$

(?) $\forall\varepsilon>0\ \exists\delta>0$ t.c. $\forall x,y\in(a,b)\ |x-y|<\delta\Rightarrow|f(x)-f(y)|<\varepsilon$

(∗) $\to\ \forall\varepsilon>0\ \exists\delta_1>0$ t.c. $\forall x,y\in(a,c]\ |x-y|<\delta_1\Rightarrow\underline{|f(x)-f(y)|<\frac{\varepsilon}{2}}$

$\exists\delta_2>0$ t.c. $\forall x,y\in[c,b)\ |x-y|<\delta_2\Rightarrow|f(x)-f(y)|<\frac{\varepsilon}{2}$

PRENDO $\delta=\min(\delta_1,\delta_2)$

$\forall(x,y)\in(a,b)$ SE $|x-y|<\delta$ ALLORA

$x,y \le c \qquad |f(x)-f(y)| < \frac{\varepsilon}{2}$ PER $(*)$

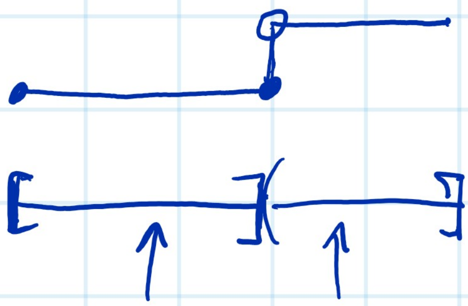
$x,y \ge c \qquad |f(x)-f(y)| < \frac{\varepsilon}{2}$ PER $(\star)$

$x<c<y \qquad |x-c| < \delta \qquad |y-c| < \delta$

$$|f(x)-f(y)| = |f(x)-f(c)+f(c)-f(y)| \le$$

$$\le \overbrace{|f(x)-f(c)|} + \overbrace{|f(c)-f(y)|} \le$$

$$\le \underset{(*)}{\frac{\varepsilon}{2}} + \underset{(\star)}{\frac{\varepsilon}{2}} = \varepsilon$$



2

$f(x) = \boxed{\frac{\sin x}{x}} \qquad \underbrace{(0,+\infty)}_{[0,+\infty)} \qquad \mathbb{R} - \{0\}$

$$F(x) = \begin{cases} f(x) & \text{SE } x \in (0,+\infty) \\ 1 & \text{SE } x = 0 \end{cases}$$

$F(x)$ È CONT. SU $[0,+\infty)$, $F(x) \to 0$ PER $x \to +\infty$

F(x) È U.C. PER T.2.

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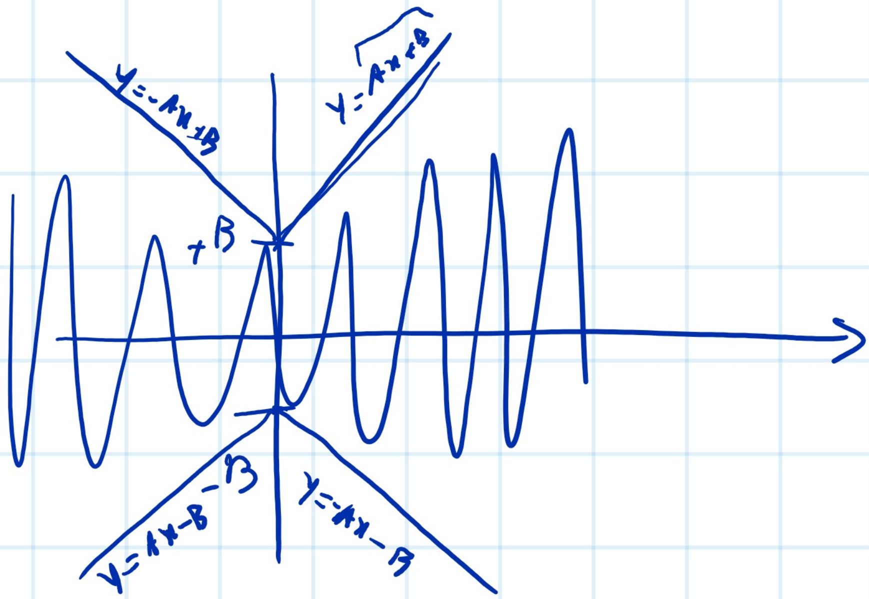
3 $\quad f(x) = \sqrt[3]{x + x^5}$ SU $[1, +\infty)$

$$f(x) \approx \boxed{x^{\frac{5}{3}}}$$

$\forall A, B > 0 \quad \frac{x^{\frac{5}{3}}}{Ax+B} \xrightarrow{x \to +\infty} +\infty$

$\nexists$ A, B > 0 t.c. $|f(x)| \leq A|x| + B$

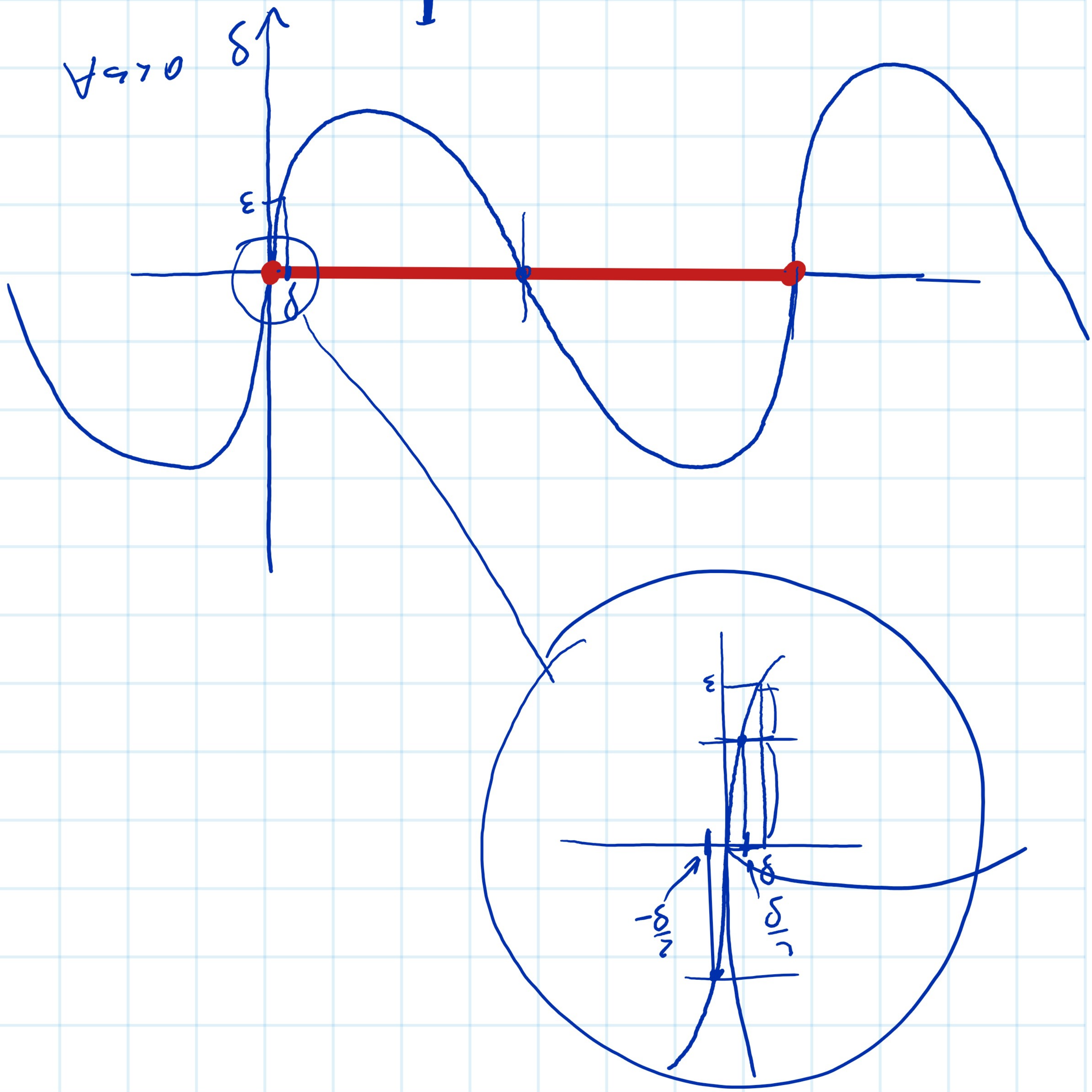


4

$f(x) = \cancel{\sqrt[3]{\sin x}}$ $\quad$ SU $\mathbb{R}$

I

$\forall \varepsilon > 0$



$\boxed{T.\ 9}$ DATA $f:\mathbb{R}\to\mathbb{R}$ CONTINUA E PERIODICA DI PERIODO $T>0$.

ALLORA f È U.C. SU $\mathbb{R}$

$\boxed{DIM}$ (1) $\forall \varepsilon>0\ \exists\delta>0$ t.c. $\forall x,y\in\mathbb{R}\ |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$

$\exists\delta>0$ t.c. $\forall x,y\in[0,2T]\ |x-y|<\delta \Rightarrow$

$\Rightarrow |f(x)-f(y)|<\varepsilon$

$x,y\ \overbrace{[kT,(k+1)T]}$

$|f(x)-f(y)| = |f(x-kT)-f(y-kT)| \leq \varepsilon$

$[0,T]\subset[0,2T]$

$(k-1)T < \boxed{x<kT<y} < (k+1)T$

$0 < x < T < y < 2T$

$|f(x)-f(y)| = |f(x-(k-1)T)-f(y-(k-1)T)|$

$\leq \varepsilon$

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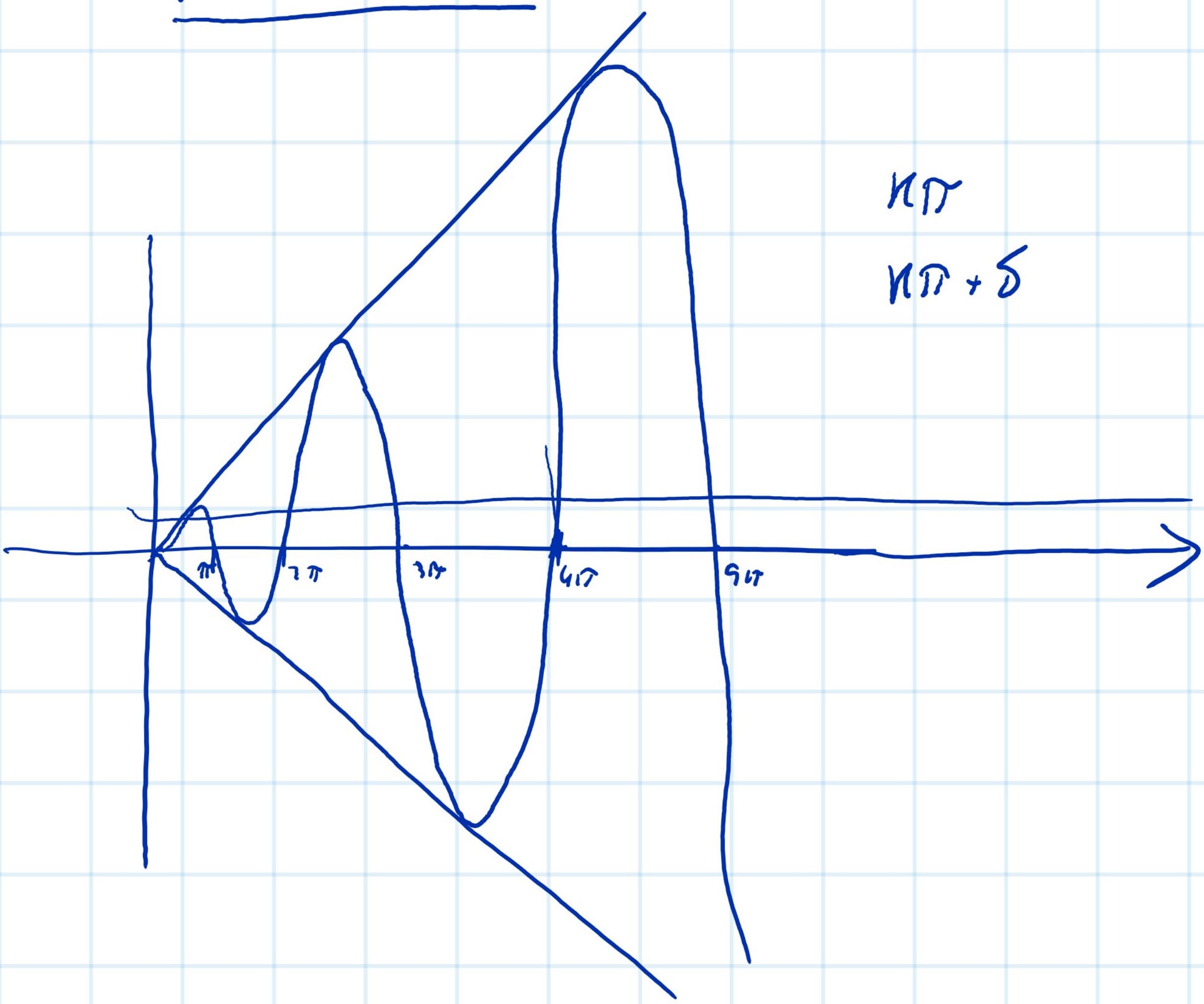
$f(x) = x \sin x$



$n\pi$

$n\pi + \delta$

$f(2n\pi) = 2n\pi \cdot \sin(2n\pi) = 0$

$f(2n\pi+\delta) = (2n\pi+\delta) \cdot \sin(2n\pi+\delta) =$

$= \underbrace{(n\pi+\delta)} \cdot \underbrace{\sin\delta} \to +\infty$

$a_n = 2n\pi \quad b_n = 2n\pi + \frac{\delta}{2}$