Esercitazioni di Analisi 1

Exe 14

13 Dicembre 2022

PROBLEMI PROPOSTI

[.... DALLA LEZIONE SCORSA]

DIRE SE LE FUNZIONI SEGUENTI SONO UNIFORMEMENTE CONTINUE E/O LIPSCHITZIANE NEGLI INSIEMI INDICATI A FIANCO:

⋮

5) $f(x) = \underbrace{\left(\arctan \frac{1}{x}\right) \cdot \sin x}_{(a)}$ SU $(0,+\infty)$ [VARIANTI: $\underbrace{\boxed{(\arctan x)\left(\sin \frac{1}{x}\right)}}_{(b)}$, $\underbrace{\boxed{(\arctan x)(\sin x)}}_{(c)}$, $\underbrace{\boxed{\left(\arctan \frac{1}{x}\right)\left(\sin \frac{1}{x}\right)}}_{(d)}$

⋮

7) $f(x) = |\sin x|^{x}$ SU $[1,100]$, $[1,+\infty)$

8) $f(x) = |\sin x|^{\frac{1}{x}}$ SU $[1,100]$, $[1,+\infty)$

9) $f(x) = d(x,\mathbb{Z})$ SU $\mathbb{R}$.

10) SIA $f:\mathbb{R}\to\mathbb{R}$ TALE CHE $\boxed{f(\ln n)=0}$ $\forall n\in\mathbb{N}-\{0\}$. DIRE SE $\lim_{x\to+\infty} f(x)=0$ NEL CASO CHE f SIA: (a) CONTINUA, (b) UNIFORMEMENTE CONTINUA, (c) LIPSCHITZIANA.
QUANDO È POSSIBILE, STIMARE L'ORDINE DI INFINITESIMO.

5a

$$f(x) = \left(\arctan \frac{1}{x}\right) \cdot (\sin x) \qquad \downarrow (0, +\infty)$$

$\boxed{\text{U.C.}}$

$$\lim_{x \to 0^+} \arctan \frac{1}{x} \cdot \sin x = 0$$

$\boxed{\text{LIP. ?}}$

$$F(x) = \begin{cases} 0 & \text{se } x = 0 \\ f(x) & \text{se } x > 0 \end{cases}$$

$$F(x) \qquad [0, +\infty)$$

$$\uparrow$$

$$\lim_{x \to +\infty} F(x) = 0$$

$$|f'(x)| = \left|\left(\arctan \frac{1}{x} \cdot \sin x\right)'\right| = \left| \frac{1}{1 + \left(\frac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2}\right) \cdot \sin x + \arctan \frac{1}{x} \cdot \cos x \right| =$$

$$= \left| \underbrace{- \frac{1}{x^2 + 1} \cdot \sin x + \cos x \arctan \frac{1}{x}}_{} \right| \leq$$

$$\leq \boxed{1 + \frac{\pi}{2}}$$

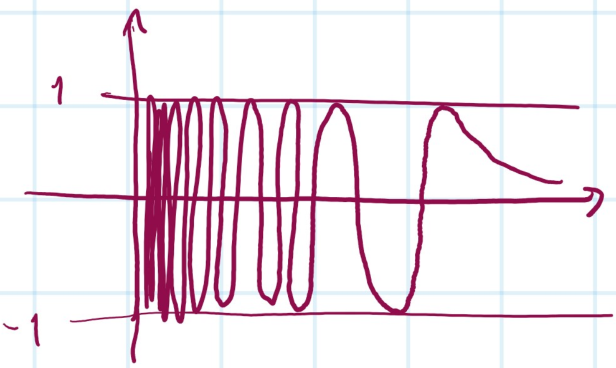
$\exists L>0 \quad \forall x,y\in(0,+\infty) \quad \boxed{|f(x)-f(y)|\le L\cdot|x-y|}$

$$\left|\frac{f(x)-f(y)}{x-y}\right|\le L$$

$\uparrow$

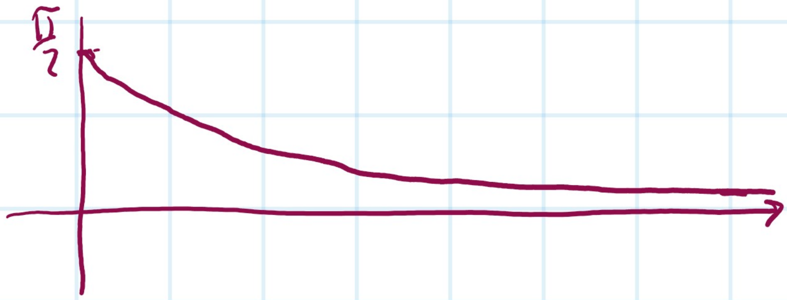
$\exists\, \xi\in(x,y)$ t.c. $\left|\frac{f(x)-f(y)}{x-y}\right| = |f'(\xi)|$ $\uparrow$

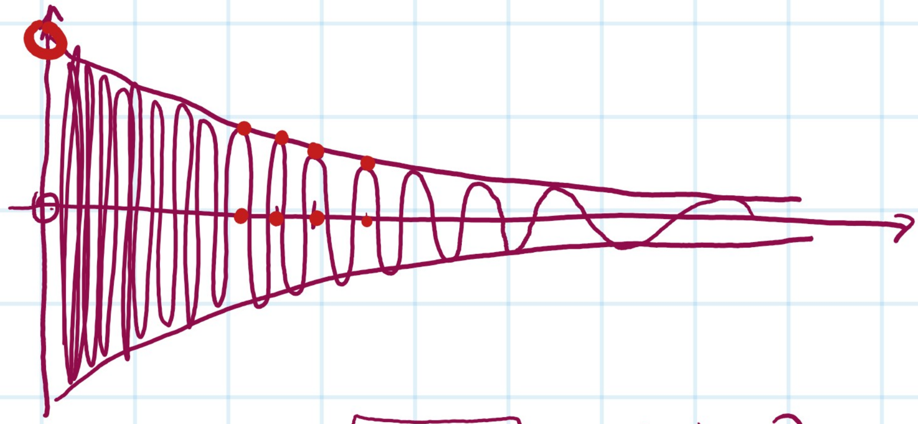
5a $\quad f(x)=\left(\arctan\frac{1}{x}\right)\cdot\left(\sin\frac{1}{x}\right) \qquad (0,+\infty)$



$\boxed{x>0}$

$\boxed{\arctan x+\arctan\frac{1}{x}=\frac{\pi}{2}}$





$\overbrace{f(a_n) \to 0}$ $\quad f(b_n) \to \frac{\pi}{2}$

$\left(\text{arctan} \frac{1}{x}\right) \cdot \sin\left(\frac{1}{x}\right)$

$a_n, b_n \to 0^+$

$a_n = \frac{1}{2\pi n}$ $\quad f(a_n) = \overbrace{\text{arctan}(2\pi n)} \cdot \overbrace{\sin(2\pi n)} = 0$

$b_n = \frac{1}{2\pi n + \frac{\pi}{2}}$

$$f(b_n) = \text{arctan}\left(2\pi n + \frac{\pi}{2}\right) \cdot \overbrace{\sin\left(\cancel{2\pi n} + \frac{\pi}{2}\right)}^{=1} =$$

$$= \text{arctan}\left(2\pi n + \frac{\pi}{2}\right) \longrightarrow \frac{\pi}{2}$$

NO U.C.

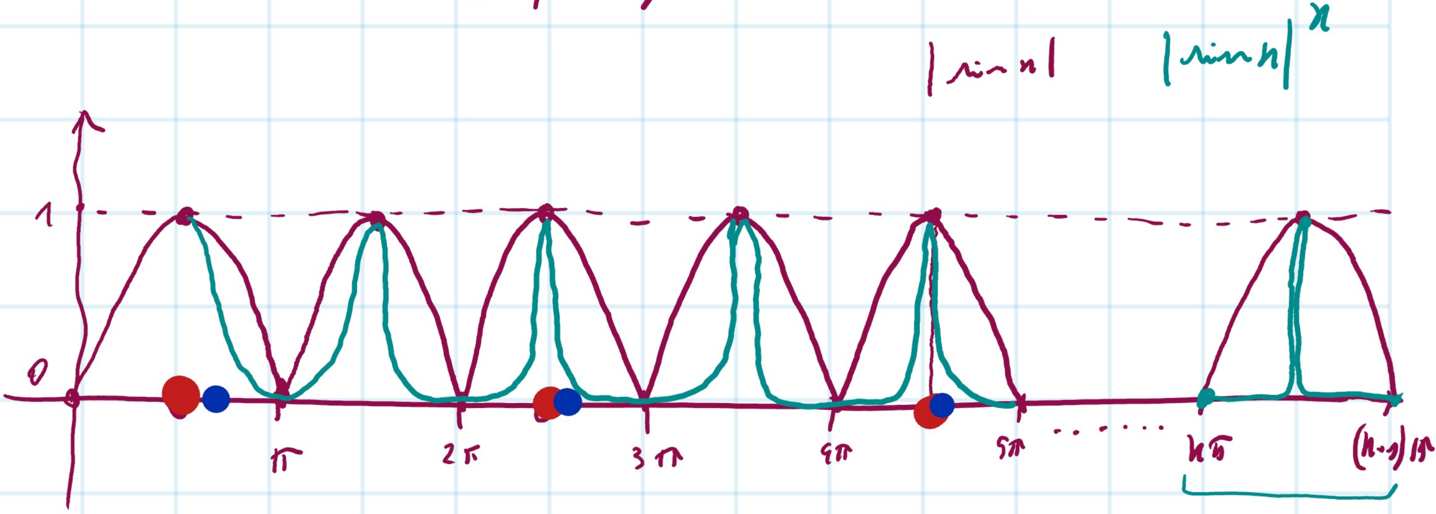
NO LIP.

$\boxed{\text{T.}}$ $\boxed{\text{DATA } f: A \to \mathbb{R} \text{ U.C. SU } A}$ ALLORA $\exists F: \bar{A} \to \mathbb{R}$ CONTINUA SU $\bar{A}$ T.C. $F(x)\big|_A = f(x)$

7) $f(x)=|\sin x|^{x}$ $\quad [1,100]$

SI PER A.C.

$[1,+\infty)$



$\exists^{no}$ a_n, b_n t.c. $|a_n-b_n|\to 0$ MA $\exists\, \varepsilon_0>0$ T.C. $|f(a_n)-f(b_n)|\geq\varepsilon_0$

$a_n=\frac{\pi}{2}+2n\pi \qquad f(a_n)=\left|\sin\left(\frac{\pi}{2}+2n\pi\right)\right|^{\frac{\pi}{2}+2n\pi}=$

$=1$

$\boxed{\frac{1}{\sqrt{n}}}$

(?) $\quad b_n=a_n+\left(\frac{1}{n}\right) \qquad f(b_n)=f\left(a_n+\frac{1}{n}\right)=$

$$= \left|\sin\left(\frac{\pi}{2} + \cancel{2n\pi} + \frac{1}{n}\right)\right|^{\frac{\pi}{2}+2n\pi+\frac{1}{n}} =$$

$$= \left|\cos\left(-\frac{1}{n}\right)\right|^{\frac{\pi}{2}+2n\pi+\frac{1}{n}} =$$

$$= \left(\cos\frac{1}{n}\right)^{\frac{\pi}{2}+2n\pi+\frac{1}{n}} = \boxed{e^{\overbrace{\left(\frac{\pi}{2}+2n\pi+\frac{1}{n}\right)\cdot\ln\left(1+\underbrace{\left(\cos\frac{1}{n}-1\right)}\right)}^{(*)}}}$$

$$(*) \sim \overbrace{\left(\frac{\pi}{2}+\underbrace{2\pi n}+\frac{1}{n}\right)}\cdot\left(\cos\frac{1}{n}-1\right) \approx 2\pi n\cdot\left(-\frac{1}{2n^2}\right) \to 0$$

n

$$e^{\overbrace{\left(\frac{\pi}{2}+\underbrace{2n\pi}+\frac{1}{\sqrt{n}}\right)\cdot\ln\left(1+\underbrace{\left(\cos\frac{1}{\sqrt{n}}-1\right)}\right)}^{(*)}}$$

$$(*) \approx 2\pi n\cdot\left(\cos\frac{1}{\sqrt{n}}-1\right) \approx 2\pi n\cdot\left(-\frac{1}{2n}\right) = -\pi$$

$$\underbrace{\left|f(b_n)-f(a_n)\right|} \to \left|e^{-\pi}-1\right| =$$

$$= 1-\frac{1}{e^{\pi}} \geq \frac{2}{3}$$

$$\underbrace{\text{DEF. in } n \quad \left|f(b_n)-f(a_n)\right| > \frac{1}{2}}$$

$\varepsilon = \frac{1}{2}$ $\quad \underline{\forall \delta > 0}$ $\quad$ Prendo n_0 t.c. $\underbrace{|b_{n_0} - a_{n_0}| < \delta}$

$$|f(b_{n_0}) - f(a_{n_0})| > \frac{1}{2}$$

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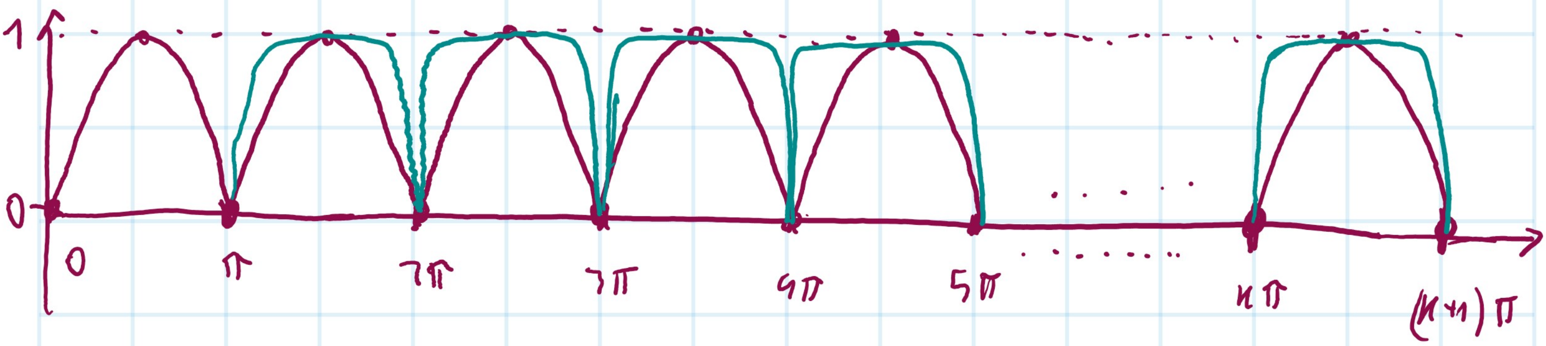
$$f(x) = |\sin x|^{\frac{1}{x}}$$

$|\sin x|$ $\qquad |\sin x|^{\frac{1}{x}}$



$$a_n = 2n\pi \qquad \boxed{f(a_n)} = |\sin 2n\pi|^{\frac{1}{2n\pi}} = 0^{\frac{1}{2n\pi}} = 0$$

$$b_n = 2n\pi + \frac{1}{n}$$

$$\underbrace{f(b_n)} = \left|\sin\left(\cancel{2n\pi} + \frac{1}{n}\right)\right|^{\frac{1}{2n\pi + \frac{1}{n}}} = \left(\sin \frac{1}{n}\right)^{\frac{1}{2n\pi + \frac{1}{n}}} =$$

$$= e^{\overbrace{\frac{1}{\underbrace{2n\pi}+\underbrace{\frac{1}{n}}} \cdot \ln\left(\sin \frac{1}{n}\right)}^{(*)}} \longrightarrow e^0 = 1$$

$$(*) \approx \frac{1}{2n\pi} \cdot \ln\left(\sin\frac{1}{n}\right) =$$

$$= \overbrace{\frac{1}{2n\pi} \cdot \frac{1}{\sin\frac{1}{n}}} \cdot \left(\sin\frac{1}{n}\right) \cdot \ln\left(\sin\frac{1}{n}\right) =$$

$$= \frac{1}{2\pi} \cdot \underbrace{\boxed{\frac{\frac{1}{n}}{\sin\frac{1}{n}}}}_{\downarrow\, 1} \cdot \overbrace{\left(\sin\frac{1}{n}\right) \ln\left(\sin\frac{1}{n}\right)} \to \frac{1}{2\pi} \cdot 1 \cdot 0 =$$

$$= 0$$

$$\overline{x \ln x} = \frac{-\ln\frac{1}{x}}{\frac{1}{x}} = -\frac{\ln y}{y} \to 0$$

$$\left|f(b_n) - f(a_n)\right| \to 1$$

$$|b_n - a_n| = \frac{1}{n} \to 0$$

NO U.C.

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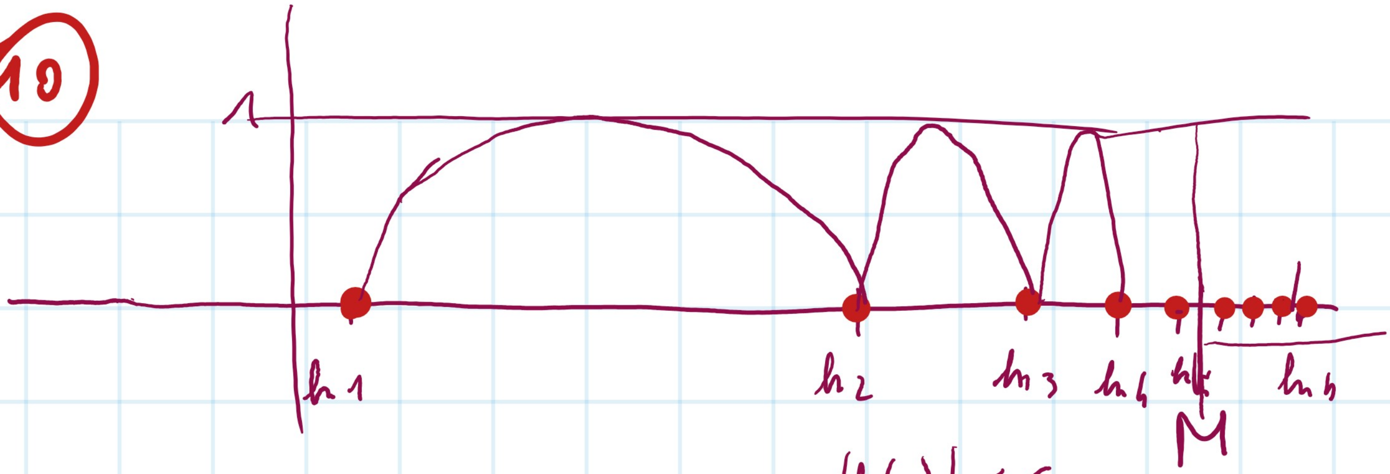

? $\forall \varepsilon > 0 \quad \exists M > 0$ t.c. $x > M \Rightarrow |f(x)| < \varepsilon$

(9.1) $\forall \varepsilon > 0 \quad \exists \delta > 0$ t.c. $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

$M > 0$ t.c. $n > M$ $\quad |\ln(n+1) - \ln n| < \delta$

$\hookrightarrow = \ln\left(1 + \frac{1}{n}\right) \to 0$

$\forall x > M$ PRENDO n t.c. $|\ln(n) - x| < \delta$

$$|f(x) - \underbrace{f(\ln n)}_{=0}| < \varepsilon \longleftrightarrow |f(x)| < \varepsilon$$

CASO LIPSCHITZ