Esercitazioni di Analisi 1

Exe 15

16 Dicembre 2022

ESERCIZI PROPOSTI (PRESI DAL CAPITOLO 4 DELL'ESERCIZIARIO)

Calcolare la derivata di $f(x)$ nei seguenti casi:

8 $f(x) = \ln\left|\ln \frac{1}{x^2+1}\right|$

7 $f(x) = \frac{1}{\sqrt{x^2+x+2} - \sqrt{x^2+x+1}}$

9 $f(x) = \log_x\left(x^2+1\right)$

16 $f(x) = \left(1+\frac{1}{x}\right)^x$

12 $f(x) = \arctan(\ln x) + \arctan(\log_x e)$

Calcolare la derivata 101-esima di $f(x)$ nei seguenti casi:

17 $f(x) = \sin 2x$

18 $f(x) = xe^x$

19 $f(x) = \frac{1}{x^2}$

20 $f(x) = x^2e^x$

21 $f(x) = \frac{x^{101}}{x-1}$

22 $f(x) = e^x\cos x$

Calcolare, se esiste, $f'(0)$ nei seguenti casi:

25 $f(x) = \begin{cases} x + 2x^2 \sin\frac{1}{x} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$

26 $f(x) = \begin{cases} x^2 & \text{se } x \in \mathbf{Q} \\ -x^2 & \text{se } x \in \mathbf{R} - \mathbf{Q} \end{cases}$

Calcolare il valore della derivata della funzione inversa di $f(x)$, nel punto y_0 indicato a fianco, nei seguenti casi:

27 $f(x) = x + e^x, \quad y_0 = 1$

29 $f(x) = x^7 + x^3 + 8, \quad y_0 = 10$

A1 DIMOSTRARE L'IDENTITÀ: $\arctan x + \arctan \frac{1}{x} = \begin{cases} -\frac{\pi}{2} & \text{SE } x<0 \\ \frac{\pi}{2} & \text{SE } x>0 \end{cases}$

99 Prima indovinare e poi dimostrare la formula per calcolare la derivata n-esima del prodotto $f(x)g(x)$.

A2 TROVARE IL MAX. DI $f(x) = x^5 - \sqrt{5x} + \sin \frac{2022\pi}{x+1}$ PER $x \in [0,5]$.

$f(0)=0$

4 Sia $f : (-1, 1) \to \mathbf{R}$ derivabile su tutto $(-1, 1)$ e tale che $f'(0) = m > 0$. Quali delle seguenti affermazioni su f sono sicuramente vere?
(a) se $\delta > 0$ è sufficientemente piccolo allora $f(-\delta) < 0 < f(\delta)$;
(b) se $\delta > 0$ è sufficientemente piccolo allora f è crescente su $(-\delta, \delta)$;
(c) $f'(x) \to m$ per $x \to 0$.

A solo **(a)** B solo **(a)** e **(b)** C solo **(c)** D tutte E solo **(a)** e **(c)** F nessuna

5 Sia $f : (-1, 1) \to \mathbf{R}$ tale che per ogni $x \in (-1, 1)$ sia $x^2 \leq f(x) \leq 2x^2$. Quali delle seguenti affermazioni sono sicuramente vere?
(a) $f(x)$ assume valore minimo per $x = 0$;
(b) esiste $\delta > 0$ tale che f decresce su $(-\delta, 0)$ e cresce su $(0, \delta)$;
(c) f è derivabile per $x = 0$ e si ha $f'(0) = 0$.

A solo **(a)** e **(b)** B nessuna C solo **(a)** e **(c)** D solo **(a)** E solo **(c)** F tutte

SOLUZIONI

8 $f(x) = \ln\left|\ln\dfrac{1}{x^2+1}\right| = \ln\left|-\ln(x^2+1)\right| = \ln\left(\ln(x^2+1)\right)$

$$f'(x) = \left(\ln(\ln(x^2+1))\right)' = \frac{1}{\ln(x^2+1)}\cdot\frac{1}{x^2+1}\cdot 2x$$

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7 $f(x) = \dfrac{1}{\sqrt{x^2+x+2}-\sqrt{x^2+x+1}} = \sqrt{x^2+x+2}+\sqrt{x^2+x+1}$

$$f'(x) = (\ldots)' = \frac{2x+1}{2\sqrt{x^2+x+2}} + \frac{2x+1}{2\sqrt{x^2+x+1}}$$

16 $f(x) = \left(1+\dfrac{1}{x}\right)^x = e^{x\ln\left(1+\frac{1}{x}\right)} = e^{x\left(\ln(x+1)-\ln x\right)}$

$$f'(x) = \left(1+\frac{1}{x}\right)^x\cdot\left(\underbrace{\ln(x+1)-\ln x}_{\downarrow} + \overbrace{x\left(\frac{1}{1+x}-\frac{1}{x}\right)}^{\frac{\cancel{x}-\cancel{x}-1}{(1+x)\cdot x}}\right) =$$

$$= \underbrace{\left(1+\frac{1}{x}\right)^x\cdot\left(\ln\left(1+\frac{1}{x}\right)-\frac{1}{x+1}\right)}$$

12 $f(x) = \arctan(\ln x) + \arctan\big(\overbrace{\log_x e}^{\frac{1}{\ln x}}\big)$

$$f'(x) = \left(\arctan(\ln x) + \arctan\left(\frac{1}{\ln x}\right)\right)' = 0$$

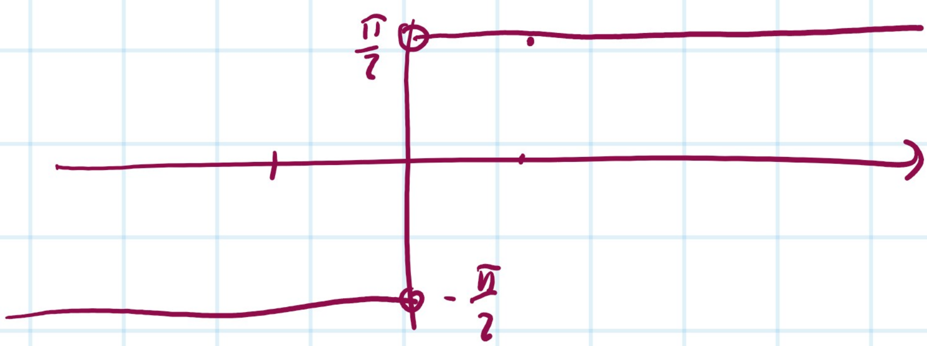
$$\arctan t + \arctan \frac{1}{t} = \begin{cases} -\frac{\pi}{2} & t<0 \\ \frac{\pi}{2} & t>0 \end{cases}$$

$$\left(\boxed{\arctan t + \arctan \frac{1}{t}}\right)' = \frac{1}{1+t^2} + \frac{1}{1+\left(\frac{1}{t}\right)^2}\cdot\left(-\frac{1}{t^2}\right) =$$

$$= \frac{1}{1+t^2} - \frac{1}{1+t^2} = 0$$

$t = 1$

$x<0$ $\quad$ $x>0$ $\quad$ $\arctan \underset{(-1)}{1} + \arctan \underset{(-1)}{1} = \frac{\pi}{2}$



$$\underbrace{(1-x)}(1+x+x^2+x^3+\dots+x^n) = \underline{1-x^{n+1}}$$

$$\left(\frac{x^{101}}{x-1}\right)^{(101)} =$$

$$1+x+x^2+\dots+x^n = \frac{1-x^{n+1}}{1-x} = \boxed{\frac{x^{n+1}-1}{x-1}}$$

$$= \left(\frac{x^{101}-1+1}{x-1}\right)^{(101)} = \left(\frac{x^{101}-1}{x-1} + \frac{1}{x-1}\right)^{(101)} =$$

$$= \left(\underbrace{\overbrace{1+x+x^2+\dots+x^{100}}}{} + \boxed{\frac{1}{x-1}}\right)^{(101)} =$$

$$= \left(\frac{1}{x-1}\right)^{(101)} = \Big(\underbrace{(x-1)^{-1}}_{f(x)}\Big)^{(101)}$$

$$f'(x) = (-1)(x-1)^{-2}$$

$$f''(x) = (-1)\cdot(-2)(x-1)^{-3}$$

$$f'''(x) = (-1)(-2)(-3)(x-1)^{-4}$$

$$\vdots$$

$$f^{(101)}(x) = (-1)(-2)\dots(-101)\cdot(x-1)^{-102} =$$

$$= -101!\cdot\frac{1}{(x-1)^{102}}$$

$$f(x) = xe^x = (0+x)e^x$$

$$f^{(101)}(x) = ?$$

$$f'(x) = (xe^x)' = e^x + xe^x = (1+x)e^x$$

$$f''(x) = \left((1+x)e^x\right)' = \left(e^x + (1+x)e^x\right) = (2+x)e^x$$

$$f^{(n)}(x) = (n+x)e^x$$

$$f^{(k)}(x) = (k+x)e^x$$

$$f^{(k+1)}(x) = \left((k+x)e^x\right)' = \left(e^x + (k+x)e^x\right) =$$

$$= (k+1+x)e^x$$

$$f^{(101)}(x) = (101+x)e^x$$

$$f(x) = x^2e^x$$

$$f^{(101)}(x) = \ldots\ldots$$

$$\left(f(x)\cdot g(x)\right)' = f'(x)g(x) + f(x)g'(x)$$

$$\left(f(x)\cdot g(x)\right)^{(n)}$$

$$(f\cdot g)' = f'g + fg'$$

$$(f\cdot g)'' = (f'g + fg')' = f''g + f'g' + f'g' + fg'' = f''g + 2f'g' + fg''$$

$$\vdots$$

$$(f\cdot g)^{(n)} = \binom{n}{0} f^{(n)} g + \binom{n}{1} f^{(n-1)} g' + \binom{n}{2} f^{(n-2)} g'' + \dots + \binom{n}{n} f \cdot g^{(n)} =$$

$$= \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}$$

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

$$(f\cdot g)^{(m)} = \sum_{k=0}^{m} \binom{m}{k} f^{(m-k)} \cdot g^{(k)}$$

$$(f\cdot g)^{(m+1)} = \left(\sum_{k=0}^{m} \binom{m}{k} f^{(m-k)} \cdot g^{(k)} \right)' =$$

$$= \sum_{k=0}^{m} \binom{m}{k} \left(f^{\overset{(m+1-k)}{(m-k+1)}} g^{(k)} + f^{\overset{(m+1-(k+1))}{(m-k)}} g^{(k+1)} \right) =$$

$$= \sum_{k=0}^{m} \binom{m}{k} f^{(m+1-k)} g^{(k)} + \sum_{p=1}^{m+1} \binom{m}{p-1} f^{(m+1-p)} g^{(p)}$$

$$= \sum_{k=0}^{m} \binom{m}{k} f^{(m+1-k)} g^{(k)} + \sum_{k=1}^{m+1} \binom{m}{k-1} f^{(m+1-k)} g^{(k)} =$$

$$= \binom{m}{0} f^{(m+1)} g + \sum_{k=1}^{m} \left[\binom{m}{k} + \binom{m}{k-1}\right] f^{(m+1-k)} g^{(k)} + \binom{m}{m} f \cdot g^{(m+1)} =$$

$$= \binom{m+1}{0} f^{(m+1)} g + \sum_{k=1}^{m} \binom{m+1}{k} f^{(m+1-k)} g^{(k)} + \binom{m+1}{m+1} f \cdot g^{(m+1)} =$$

$$\binom{m}{k} + \binom{m}{k-1} = \boxed{\frac{m!}{k!\,(m-k)!}} + \frac{m!}{(k-1)! \cdot (m+1-k)!} =$$

$$= \frac{m!}{k!\,(m+1-k)!} \overbrace{\left((m+1-\cancel{k}) + \cancel{k}\right)}^{m+1} =$$

$$\boxed{\frac{m!\,(m+1-k)}{k!\,(m+1-k)!}}$$

$$= \frac{(m+1)!}{k!\,(m+1-k)!} = \binom{m+1}{k}$$

$$= \sum_{k=0}^{m+1} \binom{m+1}{k} f^{(m+1-k)} g^{(k)}$$

$$\left(x^2 \cdot e^x \right)^{(101)} = \sum_{k=0}^{101} \binom{101}{k} \left(x^2\right)^{(101-k)} \cdot \left(e^x\right)^{(k)} =$$

$$= \sum_{k=99}^{101} \binom{101}{k} (x^2)^{(101-k)} (e^x)^{(k)} =$$

$$= \binom{101}{99} (x^2)'' \cdot (e^x)^{(99)} + \binom{101}{100} (x^2)' (e^x)^{(100)} + \binom{101}{101} x^2 (e^x)^{(101)} =$$

$$= \frac{101 \cdot 100}{\cancel{2}} \cdot \cancel{2} \cdot e^x + 101 \cdot 2x \cdot e^x + 1 \cdot x^2 \cdot e^x =$$

$$= \boxed{e^x \left(10100 + 202x + x^2 \right)}$$

25 $f(x) = \begin{cases} x + 2x^2 \sin \frac{1}{x} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$

26 $f(x) = \begin{cases} x^2 & \text{se } x \in \mathbf{Q} \\ -x^2 & \text{se } x \in \mathbf{R} - \mathbf{Q} \end{cases}$

$f'(0)$

$$f'(0) = \lim_{x \to 0} \frac{f(x) - \overset{0}{f(0)}}{x - 0} = \lim_{x \to 0} \frac{x + 2x^2 \sin \frac{1}{x}}{x} =$$

$$= \lim_{x \to 0} \left(1 + 2x \sin \frac{1}{x} \right) = 1$$

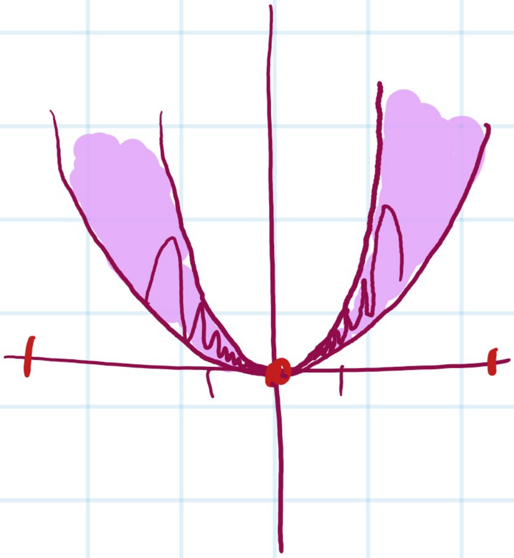
5 Sia $f : (-1,1) \to \mathbf{R}$ tale che per ogni $x \in (-1,1)$ sia $x^2 \leq f(x) \leq 2x^2$. Quali delle seguenti affermazioni sono sicuramente vere?

V **(a)** $f(x)$ assume valore minimo per $x = 0$;

F **(b)** esiste $\delta > 0$ tale che f decresce su $(-\delta, 0)$ e cresce su $(0, \delta)$;

V **(c)** f è derivabile per $x = 0$ e si ha $f'(0) = 0$.

[A] solo **(a)** e **(b)** [B] nessuna [~~C~~] solo **(a)** e **(c)** [D] solo **(a)** [E] solo **(c)** [F] tutte



$$f'_+(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} =$$

$$= \lim_{x \to 0^+} \frac{f(x)}{x} = 0$$

$$\boxed{x^2 \leq f(x) \leq 2x^2}$$

$$\underset{\downarrow \atop 0}{x} \leq \underset{\downarrow \atop 0}{\boxed{\frac{f(x)}{x}}} \leq \underset{\downarrow \atop 0}{2x}$$

$$\underset{\downarrow \atop 0}{x} \geq \underset{\downarrow \atop 0}{\frac{f(x)}{x}} \geq \underset{\downarrow \atop 0}{2x}$$

$$f(x) = \begin{cases} \frac{3}{2}x^2 + \frac{x^2}{2}\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

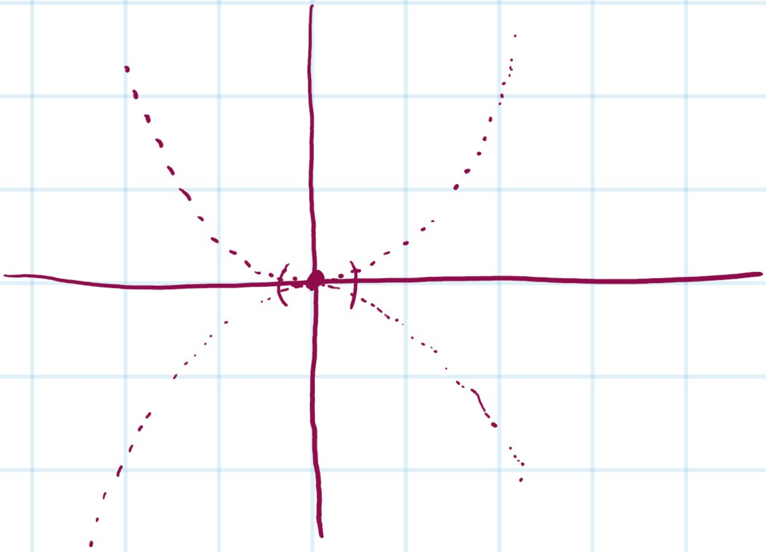
$$x^2 = \frac{3}{2}x^2 - \frac{x^2}{2} \leq \frac{3}{2}x^2 + \frac{x^2}{2}\sin\frac{1}{x} \leq \frac{3x^2}{2} + \frac{x^2}{2} = \boxed{2x^2}$$

$$f'(x) = \begin{cases} 3x + x \cdot \sin\frac{1}{x} - \frac{1}{2}\cos\frac{1}{x} & \\ 0 & x = 0 \end{cases}$$

$$\left(\frac{3}{2}x^2 + \frac{1}{2}x^2\sin\frac{1}{x}\right)' = 3x + x \cdot \sin\frac{1}{x} + \frac{1}{2}x^2 \cdot \cos\frac{1}{x} \cdot \left(-\frac{1}{x^2}\right) =$$

$$= \boxed{3x + x\sin\frac{1}{x}} - \frac{1}{2}\cos\frac{1}{x}$$

$$f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ -x^2 & x \in \mathbb{R} - \mathbb{Q} \end{cases}$$



$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{f(x)}{x} = 0$$

$$-\frac{x^2}{x} \leq \frac{f(x)}{x} \leq \frac{x^2}{x}$$

$$\underset{\substack{\downarrow \\ 0}}{-x} \leq \underset{\substack{\downarrow \\ 0}}{\frac{f(x)}{x}} \leq \underset{\substack{\downarrow \\ 0}}{x}$$

29 $f(x) = x^7 + x^3 + 8, \quad y_0 = 10$

y

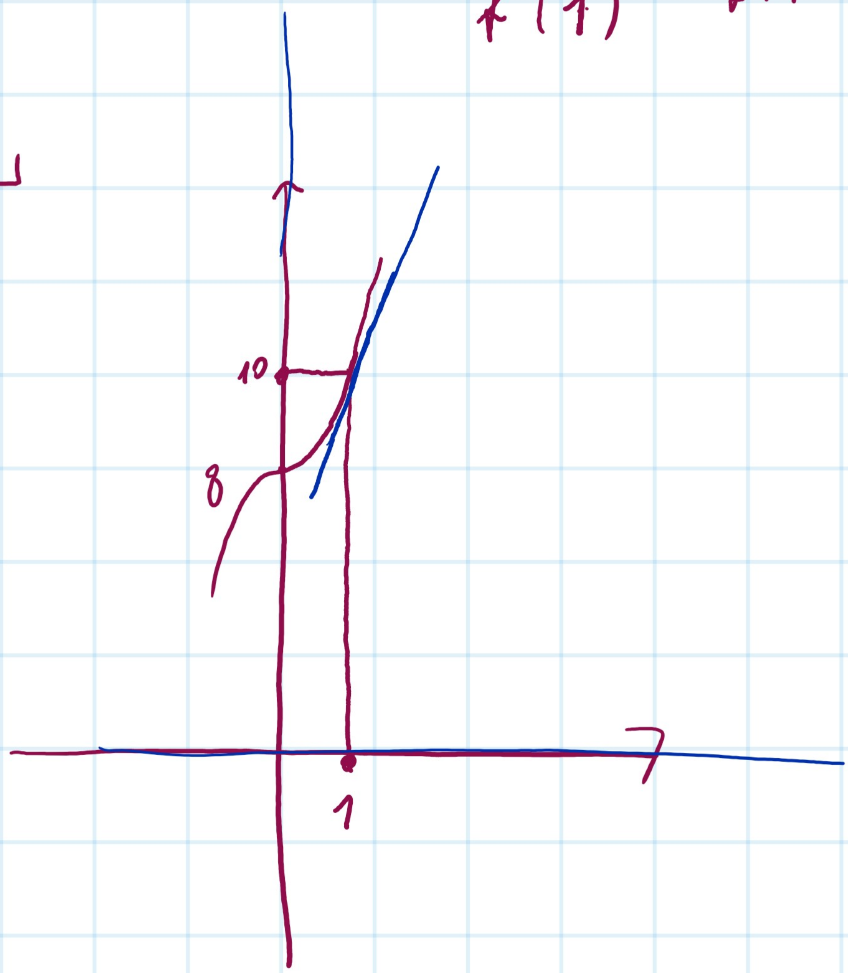
$f'(x) = 7x^6 + 3x^2$

$(f^{-1})'(10) = \frac{1}{f'(1)} = \frac{1}{7+3} = \frac{1}{10}$

$y = x^7 + x^3 + 8$

$y = f(x)$

$f^{-1}(y) = x$



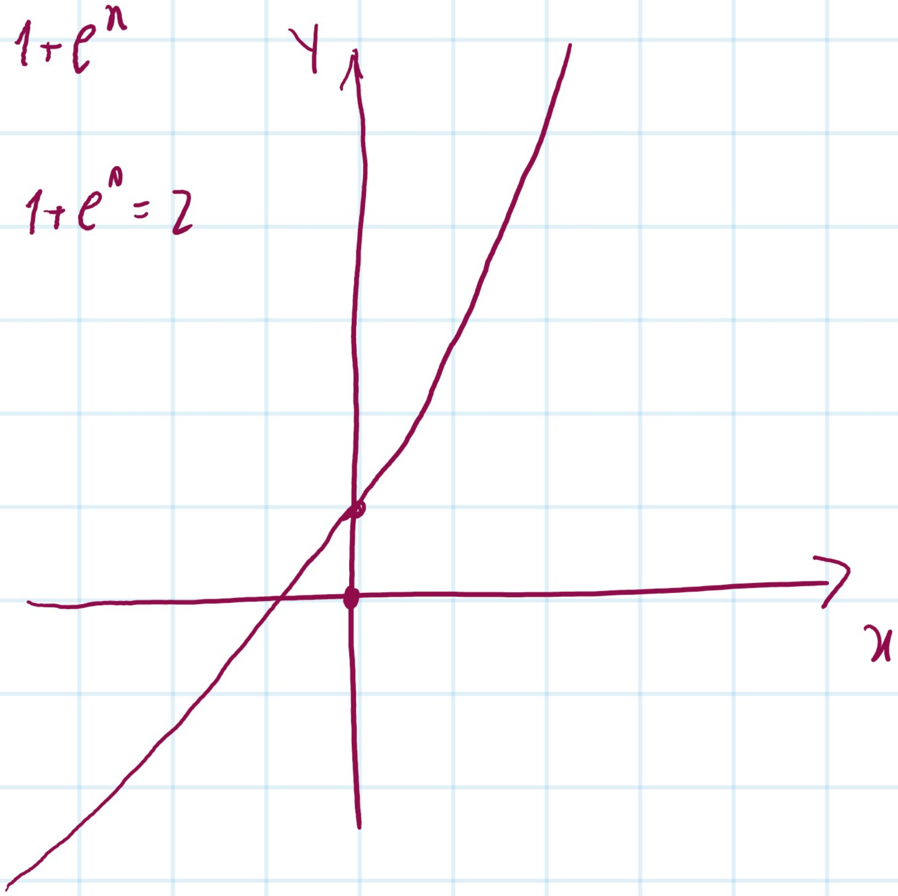
$f(x) =$

27 $f(x) = x + e^x, \quad \underline{y_0 = 1}$

$$\left(f^{-1}\right)'(1) = \frac{1}{f'(0)} = \frac{1}{2}$$

$f'(x) = 1 + e^x$

$f'(0) = 1 + e^0 = 2$



e $f(0)=0$

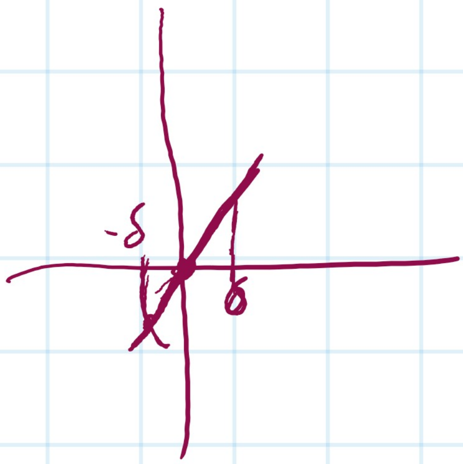
4 Sia $f:(-1,1)\to\mathbf{R}$ derivabile su tutto $(-1,1)$ e tale che $f'(0)=m>0$. Quali delle seguenti affermazioni su f sono sicuramente vere?

V **(a)** se $\delta>0$ è sufficientemente piccolo allora $f(-\delta)<0<f(\delta)$;

F **(b)** se $\delta>0$ è sufficientemente piccolo allora f è crescente su $(-\delta,\delta)$;

F **(c)** $f'(x)\to m$ per $x\to 0$.

$\boxed{\cancel{\text{A}}}$ solo **(a)** $\boxed{\text{B}}$ solo **(a)** e **(b)** $\boxed{\text{C}}$ solo **(c)** $\boxed{\text{D}}$ tutte $\boxed{\text{E}}$ solo **(a)** e **(c)** $\boxed{\text{F}}$ nessuna



$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} =$$

$\exists\,\delta>0$ t.c. $\dfrac{f(x)-f(0)}{x-0} > \dfrac{m}{2}$

per $x\in[-\delta,\delta]-\{0\}$

$x<0$

$$\frac{f(x)}{x} > \frac{m}{2}$$

$$f(x) < \frac{m}{2}x$$

$$f(-\delta) < \frac{m}{2}\delta < 0$$

$$\frac{f(x)}{x} > \frac{m}{2}$$

$$f(\delta) > \frac{m}{2}\delta > 0$$