Esercitazioni di Analisi 1

Exe 16

20 Dicembre 2022

ESERCIZI PROPOSTI (PRESI DAL CAPITOLO 4 DELL'ESERCIZIARIO)

1) DIMOSTRARE IL SEGUENTE:

Teorema 73 [Sviluppi delle funzioni elementari] Per ogni n intero positivo valgono per $x \to 0$ le seguenti identità:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots + \frac{x^n}{n!} + O\left(x^{n+1}\right) \tag{299}$$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + O\left(x^{2n+3}\right) \tag{300}$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + O\left(x^{2n+2}\right) \tag{301}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + (-1)^n \frac{x^{n+1}}{n+1} + O\left(x^{n+2}\right) \tag{302}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots + x^n + O\left(x^{n+1}\right) \tag{303}$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} + O\left(x^{2n+3}\right) \tag{304}$$

$$(1+x)^\alpha = 1 + \alpha x + \binom{\alpha}{2} x^2 + \binom{\alpha}{3} x^3 + \ldots + \binom{\alpha}{n} x^n + O\left(x^{n+1}\right) \tag{305}$$

dove nell'ultima $\alpha \in \mathbf{R}$ e abbiamo utilizzato la notazione

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)}{n!}.$$

CALCOLARE I SEGUENTI LIMITI:

58 $$\lim_{x \to 0} \frac{\ln\left(1+x^2\right) - x \arctan x}{\sin^2 x^2}$$

61 $\displaystyle\lim_{x\to 0} \frac{3\sin x - 3x\cos x}{\ln(1+3x) - 3\ln(1+x) + 3x^2}$

66 $\displaystyle\lim_{x\to 0} \frac{3\sin x - \sqrt{3}\sin\left(x\sqrt{3}\right)}{\arctan x - \arctan 2x + x}$

67 $\displaystyle\lim_{x\to 0} \frac{2x^5 + x^7\cos\frac{1}{x}}{\sin x\cos x + \frac{2}{3}x^3 - x}$

72 $\displaystyle\lim_{x\to 0} \frac{(x\cos x - \sin x)\left(x^2 - \sin x^2\right) + \frac{1}{18}x^9}{x^{11}}$

74 $\displaystyle\lim_{x\to +\infty} \frac{x^4\sin\frac{1}{x^3} - 2x^3\left(1 - \cos\frac{1}{x}\right)}{xe^{\frac{1}{x^2}} - x - \ln(1+x) + \ln x}$

79 $\displaystyle\lim_{x\to 0^+} \frac{e^{-\frac{1}{x^2}} + 2 - \cos(x - \sin x) - \sqrt[72]{1+x^6}}{x^\alpha}$

81 $\displaystyle\lim_{x\to 0^+} \frac{x^x - \cos x - x\ln x}{x^\alpha\cdot|\ln x|^\beta}$

83 $\displaystyle\lim_{x\to 0^+} \frac{\arctan\left(x + x^{2017}\right) - \arctan x - x^{2017}}{x^\alpha}$

CALCOLARE $f^{(n)}(0)$ NEL CASO SEGUENTE:

90 $f(x) = \left(\ln\sqrt{1-x^3} + x(1-\cos x)\right)\cdot\left(\sin\left(x^2+x^4\right) - \sin x^2\right)$, con $n = 11$.

Per ciascuna delle seguenti $f(x)$ dire se $x = 0$ è un punto di estremo relativo e in caso affermativo studiarne la natura.

92 $f(x) = \sin(\ln(1+x)) - \ln(1+\sin x) - \dfrac{x^4}{12}$

SOLUZIONI

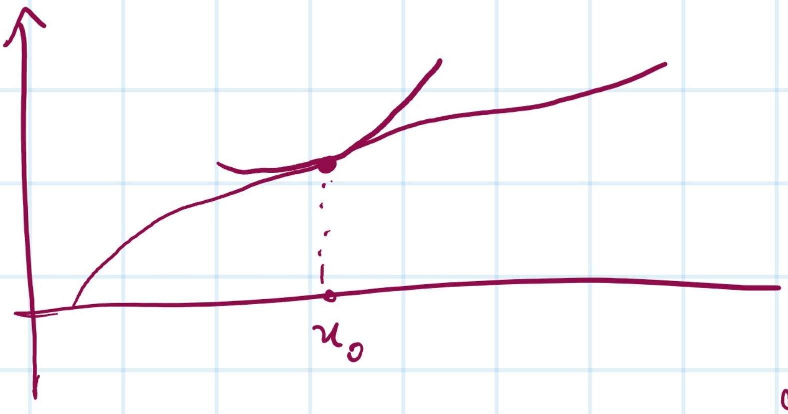
$$f(x) = P(x) + o\left((x-x_0)^n\right)$$

$$f(x) - P(x) = \boxed{\frac{f^{(n+1)}(\xi)}{(n+1)!}} \cdot (x-x_0)^{n+1}$$

$$o\left((x-x_0)^{n+1}\right)$$

$$o\left((x-x_0)^{n}\right)$$

$$o\left((x-x_0)^{n+\frac{1}{2}}\right)$$



$$\left|\frac{f(x)-P(x)}{(x-x_0)^{n+1}}\right| < C$$

$$\frac{f(x)-P(x)}{(x-x_0)^n} = \boxed{\frac{f(x)-P(x)}{(x-x_0)^{n+1}}} \cdot \underbrace{(x-x_0)}_{\to 0}$$

$$\lim_{x\to 0} \frac{e^x - x - \cos x}{x^2} = \lim_{x\to 0} \frac{\cancel{1}+\cancel{x}+\frac{x^2}{2}+o(x^2)-\cancel{x}-\left(\cancel{1}-\frac{x^2}{2}+o(x^2)\right)}{x^2}$$

$$= \lim_{x\to 0} \frac{x^2+o(x^2)}{x^2} = 1$$

$$\boxed{58}\ \lim_{x\to 0}\frac{\ln\left(1+x^2\right)-x\arctan x}{\sin^2 x^2}=$$

$$=\lim_{x\to 0}\frac{\cancel{x^2}-\frac{(x^2)^2}{2}+o(x^4)-x\left(\cancel{x}-\frac{x^3}{3}+o(x^3)\right)}{x^4}=$$

$$=\lim_{x\to 0}\frac{-\frac{1}{6}x^4+o(x^4)}{x^4}=-\frac{1}{6}$$

$$\boxed{61}\ \lim_{x\to 0}\frac{\overbrace{3\sin x-3x\cos x}^{(*)}}{\underbrace{\ln(1+3x)-3\ln(1+x)+3x^2}_{(**)}}=$$

$$(*)\quad 3\left(\cancel{x}-\frac{x^3}{6}+o(x^3)\right)-3x\left(\cancel{1}-\frac{x^2}{2}+o(x^2)\right)=$$

$$=\boxed{x^3+o(x^3)}$$

$$(*) = \cancel{(3x)} - \frac{1}{2}\cancel{(3x)^2} + \frac{1}{3}(3x)^3 + o(x^3) - 3\left(\cancel{x} - \cancel{\frac{x^2}{2}} + \frac{1}{3}x^3 + o(x^3)\right)$$

$$\boxed{\cancel{-\frac{9}{2}x^2}} \qquad \cancel{3x^2} \qquad \boxed{\cancel{\frac{3x^2}{2}}}$$

$$= 9x^3 - x^3 + o(x^3) = 8x^3 + o(x^3)$$

$$\to = \lim_{x\to 0} \frac{x^3 + o(x^3)}{8x^3 + o(x^3)} = \frac{1}{8}$$

67 $\displaystyle \lim_{x\to 0} \frac{2x^5 + \boxed{x^7 \cos \frac{1}{x}}^{\,o(x^5)}}{\underbrace{\underbrace{\sin x \cdot \cos x}_{\frac{1}{2}\sin(2x)} + \frac{2}{3}x^3 - x}_{(*)}} =$

$$\lim_{x\to 0} \frac{x^{\cancel{7}\,2} \cos\frac{1}{x}}{\cancel{x^5}} = 0$$

$$= \lim_{x\to 0} \frac{2x^5 + o(x^5)}{\frac{2}{15}x^5 + o(x^5)} = 15$$

$$(*) = \frac{1}{2}\left(\cancel{2x} - \cancel{\frac{(2x)^3}{6}} + \frac{(2x)^5}{120} + o(x^5)\right) + \cancel{\frac{2}{3}x^3} - \cancel{x} = \frac{\cancel{16}^{\,2}}{\cancel{120}_{\,15}}x^5 + o(x^5)$$

72 $\displaystyle \lim_{x\to 0} \frac{\overbrace{(x\cos x - \sin x)}^{\downarrow}\overbrace{\left(x^2 - \sin x^2\right)}^{\downarrow} + \frac{1}{18}x^9}{x^{11}} =$

$$= \lim_{x\to 0} \frac{-\cancel{\frac{1}{18}x^9} + \frac{1}{180}x^{11} + o(x^{13}) - \cancel{\frac{1}{18}x^9}}{x^{11}} = \frac{1}{180}$$

$$x\cos x - \sin x = x\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \quad\right) - \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \quad\right)$$

$$= -\frac{1}{3}x^3 + \frac{1}{30}x^5 + O(x^7)$$

$$x^2 - \sin x^2 = x^2 - \left(x^2 - \frac{(x^2)^3}{6} + \quad\right)$$

$$= \frac{x^6}{6} + O(x^{10})$$

$$\left(-\frac{1}{3}x^3 + \frac{1}{30}x^5 + O(x^7)\right)\left(\frac{x^6}{6} + O(x^{10})\right) =$$

$$-\frac{1}{18}x^9 + \frac{1}{180}x^{11} + O(x^{13}) + O(x^{13}) + O(x^{15}) + O(x^{17}) =$$

62

$$\lim_{x \to 0} \frac{(x - \arctan x)\cdot(\ln(1+x^2) - x\arctan x) + \frac{x^7}{78}}{x^3 - \sin x^3} =$$

$$x^3 - \sin x^3 \to \cancel{x^3} - \cancel{x^3} + \boxed{\frac{x^9}{6}} + O(x^9)$$

$$x - \arctan x = \left(\frac{x^3}{3} - \frac{x^5}{5} + O(x^7)\right)$$

$$\ln(1+x^2) - x\arctan x = \left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} + O(x^8)\right) - x\left(x - \frac{x^3}{3} + \frac{x^5}{5} + O(x^9)\right)$$

$$\left(\frac{x^3}{3} - \frac{x^5}{5} + O(x^7)\right)\left(-\frac{1}{6}x^4 + \frac{2}{15}x^6 + O(x^8)\right)$$

$$= -\frac{1}{18}x^7 + \frac{1}{30}x^9 + O(x^{11}) + \frac{2}{45}x^9 + O(x^{15}) + \overbrace{O(x^{13}) + O(x^{11}) + O(x^{13}) + \cdots}^{O(x^{11})}$$

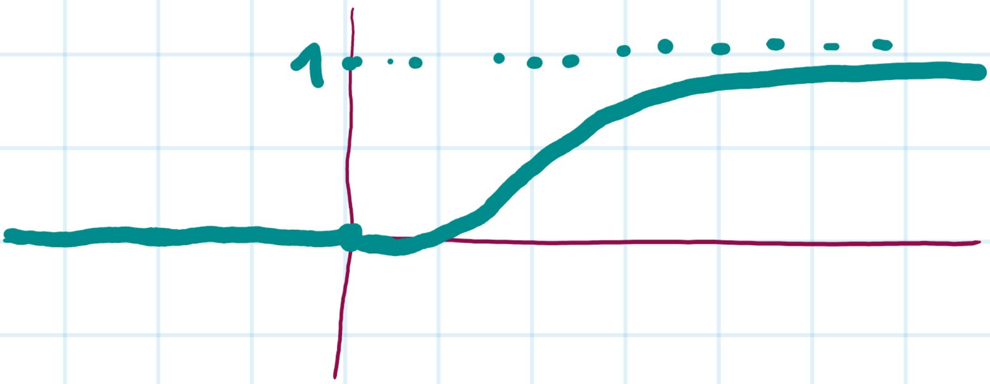
$$= \boxed{-\frac{1}{18}x^7 + \frac{7}{90}x^9 + O(x^{11})}$$

$$= \lim_{x\to 0} \frac{\frac{7}{90}x^9 + o(x^9)}{\frac{x^9}{6} + o(x^9)} = \frac{7}{15}$$

79 $$\lim_{x\to 0^+} \frac{e^{-\frac{1}{x^2}} + 2 - \cos\left(x - \sin x\right) - \sqrt[72]{1+x^6}}{x^\alpha}$$

$$f(x) = \begin{cases} 0 & x \le 0 \\ \boxed{e^{-\frac{1}{x}}} & x > 0 \end{cases}$$

$f(x) = 0 + \boxed{o(x^n)}$



$$f'(x) = \begin{cases} 0 & x \le 0 \\ \boxed{\frac{1}{x^2}}\, e^{-\frac{1}{x}} & x > 0 \end{cases}$$

$$\frac{\frac{1}{x^2}}{e^{\frac{1}{x}}}$$

$$\frac{y^2}{e^y}$$

$$f''(x) = -2\cdot\frac{1}{x^3}e^{-\frac{1}{x}} + \frac{1}{x^2}\cdot\frac{1}{x^2}e^{-\frac{1}{x}} = \left(-2\cdot\frac{1}{x^3} + \frac{1}{x^4}\right)e^{-\frac{1}{x}}$$

$\boxed{-2y^3 + y^4}$

$$\boxed{f^{(n)}(x)} \begin{cases} 0 & \boxed{x \le 0} \\ \underline{\underline{Q\left(\frac{1}{x}\right) e^{-\frac{1}{x}}}} & x > 0 \end{cases}$$

$$Q'\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right) e^{-\frac{1}{x}} + Q\left(\frac{1}{x}\right)\cdot\frac{1}{x^2} e^{-\frac{1}{x}} =$$

$$= \underbrace{\frac{1}{x^2}\left(-Q'\left(\frac{1}{x}\right) + Q\left(\frac{1}{x}\right)\right)}\, e^{-\frac{1}{x}}$$

$$\boxed{H\left(\frac{1}{x}\right)\cdot e^{-\frac{1}{x}}}$$

$$\cos y = 1 - \frac{y^2}{2} + \frac{y^4}{24} - \dots$$

x^3

$$\overline{x - \sin x} = \; = \cancel{x} - \cancel{x} + \frac{x^3}{6} + o(x^4)$$

$$\frac{\frac{1}{12}\cdot\left(\frac{1}{12}-1\right)}{2} =$$

$$2 - \underline{\cos(\overline{x - \sin x})} - \underline{\sqrt[12]{1 + x^6}}$$

$$\cancel{2} - \cancel{1} + \frac{1}{2}(x - \sin x)^2 - \frac{1}{24}(x - \sin x)^4 + O(x^{18}) - \underline{\left(\cancel{1} + \frac{1}{12}x^6 + \binom{\frac{1}{12}}{2} x^{12} + O(x^{18})\right)}$$

$$\approx \frac{1}{2}\left(\frac{x^3}{6} - \frac{x^5}{120} + o(x^7)\right)^2 - \frac{1}{24}\overline{\left(\frac{x^3}{6} - \frac{x^5}{120} + o(x^7)\right)^4} + O(x^{18}) - \underline{\left(\dots \right)}$$

$$= \frac{1}{2}\left(\cancel{\frac{x^6}{36}} - \frac{1}{360}x^8 + o(x^{10})\right) + O(x^{12}) - \cancel{\frac{1}{72}x^6} + o(x^{12}) =$$

$$= \boxed{-\frac{1}{720}x^8 + o(x^8)}$$