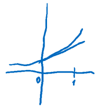
Esercitazioni di Analisi 1 - A.A. 21/22 - Callegari

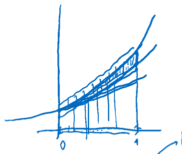
Lezione 18

17 Gennaio 2022

$f(x)=e^x$

$e^x = \boxed{1+x+\frac{x^2}{2}+\dots+\frac{x^n}{n!}} + \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}}$ $\quad 0<\xi<x$





$\sup_{x\in[0,1]} |e^x - P_n(x)| = \; \le \left|\frac{f^{(n+1)}(\xi)}{(n+1)!}\cdot x^{n+1}\right| \le \frac{1}{(n+1)!}\cdot 1\cdot e \le \frac{e}{(n+1)!}$

$0<\xi<x \quad |e^{\xi}| \le e$

$\boxed{P_n(x) = 1+x+\frac{x^2}{2}+\dots+\frac{x^n}{n!}}$

$P_n(1) = 1+1+\frac{1}{2}+\frac{1}{3!}+\dots+\frac{1}{n!} = \boxed{\sum_{k=0}^{n}\frac{1}{k!}}$

$\lim_{n\to+\infty} |e^x - P_n(x)| = 0$

$\lim_{n\to+\infty} |e - P_n(1)| = 0$

$\lim_{n\to+\infty} P_n(1) = e$

$\lim_{n\to+\infty} \sum_{k=0}^{n}\frac{1}{k!} = e$

$(1+\frac{1}{n})^n \to e$

$\left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\dots+\frac{1}{k!}\right) \to e$

$\frac{1}{k!}$

$\sum_{n=1}^{+\infty}\frac{1}{n!} = e$

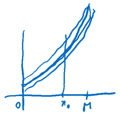
$|e^{\frac{1}{2}} - P_n(\frac{1}{2})| \to 0 \quad n\to+\infty$

$\boxed{e^x - P_n(x) = \frac{e^{\xi}}{(n+1)!}\cdot x^{n+1}}$ $\quad 0<\xi<x \quad x\in[0,M]$

$0 \le \xi \le x \le M$

$|e^x - P_n(x)| = \left|\frac{e^{\xi}}{(n+1)!}\cdot x^{n+1}\right| \le \boxed{\frac{e^M\cdot M^{n+1}}{(n+1)!}} \to 0 \quad n\to+\infty$

$\lim_{n\to+\infty} |e^{x_0} - P_n(x_0)| = 0$

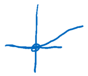


$\boxed{e^x} = S(x) = \boxed{\sum_{n=0}^{+\infty}\frac{x^n}{n!}}$

$P_n(2) \to e^2 \quad n\to+\infty$

$P_n(2) = \left(1+\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+\dots+\frac{2^n}{n!}\right)\dots \to e^2$

$\to f(x) = \begin{cases} 0 & x\le 0 \\ e^{-\frac{1}{x}} & x>0 \end{cases}$



$f'(x) = \begin{cases} 0 & x<0 \\ ? & x=0 \\ \frac{1}{x^2}e^{-\frac{1}{x}} & x>0 \end{cases}$

$\boxed{f'_-(0)=0}$

$\boxed{f'_+(0)} = \lim_{x\to0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to0^+}\frac{e^{-\frac{1}{x}}}{x} = \lim_{x\to0^+}\frac{\frac{1}{x}}{e^{\frac{1}{x}}} \overset{y=\frac{1}{x}}{=} \lim_{y\to+\infty}\frac{y}{e^y} = 0$

$P(y) = y^4 - 2y^3$

$P(\frac{1}{x})\cdot e^{-\frac{1}{x}}$

$\left(\frac{1}{x^2}e^{-\frac{1}{x}}\right)' = -\frac{2}{x^3}e^{-\frac{1}{x}} + \left(\frac{1}{x^2}\right)^2 e^{-\frac{1}{x}} = \boxed{\left(-\frac{2}{x^3}+\frac{1}{x^4}\right)e^{-\frac{1}{x}}} = P(\frac{1}{x})e^{-\frac{1}{x}}$

$\boxed{f(x) = \begin{cases} 0 & x\le 0 \\ e^{-\frac{1}{x}} & x>0 \end{cases}}$

$\boxed{f'(x) = \begin{cases} 0 & x\le 0 \\ \frac{1}{x^2}e^{-\frac{1}{x}} & x>0 \end{cases}}$

$f''(x) = \begin{cases} 0 & x<0 \\ 0 & x=0 \\ P(\frac{1}{x})e^{-\frac{1}{x}} & x>0 \end{cases}$

$\boxed{f''(x) = \begin{cases} 0 & x\le 0 \\ P_2(\frac{1}{x})e^{-\frac{1}{x}} & \end{cases}}$

$\vdots$

$\boxed{f^{(n)}(x) = \begin{cases} 0 & x\le 0 \\ P_n(\frac{1}{x})\cdot e^{-\frac{1}{x}} & x>0 \end{cases}}$

$\left(P_n(\tfrac{1}{x})\cdot e^{-\frac{1}{x}}\right)' = P_n'(\tfrac{1}{x})\cdot(-\tfrac{1}{x^2})\cdot e^{-\frac{1}{x}} + P_n(\tfrac{1}{x})\cdot e^{-\frac{1}{x}}\cdot\tfrac{1}{x^2} =$

$= \left(P_n(\tfrac{1}{x}) - P_n'(\tfrac{1}{x})\right)\frac{1}{x^2}\cdot e^{-\frac{1}{x}} =$

$= P_{n+1}(\tfrac{1}{x})\,e^{-\frac{1}{x}}$

$P_{n+1}(y) = (P_n(y) - P_n'(y))\,y^2$

$f \in C^{\infty}(\mathbb{R})$

$f^{(n)}(0) = 0 \quad \forall n$

$f''_-(0) = 0$

$\boxed{f''_+(0)} = \lim_{x\to0^+}\frac{f'(x)-f'(0)}{x-0} = \lim_{x\to0^+}\frac{\frac{1}{x^2}e^{-\frac{1}{x}}}{x} = \lim_{x\to0^+}\frac{\frac{1}{x^3}}{e^{\frac{1}{x}}} \overset{y=\frac{1}{x}}{=} \lim_{y\to+\infty}\frac{y^3}{e^y} = 0$

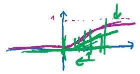
$f^{(n+1)}(x) = \begin{cases} 0 & x\le 0 \\ P_{n+1}(\frac{1}{x})e^{-\frac{1}{x}} & x>0 \end{cases}$

$f^{(n+1)}(x) = \begin{cases} 0 & x<0 \\ 0 & x=0 \\ P_{n+1}(\frac{1}{x})e^{-\frac{1}{x}} & x>0 \end{cases}$

$f^{(n+1)}_-(0) = 0$

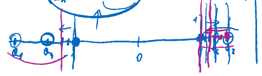
$f^{(n+1)}_+(0) = \lim_{x\to0^+}\frac{f^{(n)}(x)-f^{(n)}(0)}{x-0} = \lim_{x\to0^+}\frac{P_n(\frac{1}{x})e^{-\frac{1}{x}}}{x} = \lim_{x\to0^+}\frac{\frac{1}{x}P_n(\frac{1}{x})}{e^{\frac{1}{x}}} \overset{y=\frac{1}{x}}{=} \lim_{y\to+\infty}\frac{yP_n(y)}{e^y} = 0$

$\forall n\in\mathbb{N} \quad T_n(x) \equiv 0$



$\limsup_{n\to+\infty} a_n = \lim_{n\to+\infty}\left(\sup_{k\ge n} a_k\right) = 1$

$a_n = (-1)^n(1+\frac{1}{n})$



DEF DATA (a_n) E $\ell\in\mathbb{R}^*$ DIREMO CHE ℓ È P.T. LIMITE DI a_n SE $\exists\,(a_{n_k})$ SSUCC. DI a_n t.c. $a_{n_k}\to\ell$.

$\to \boxed{a_n = \log_2 n - \lfloor\log_2 n\rfloor}$

$n_k = 2^k$

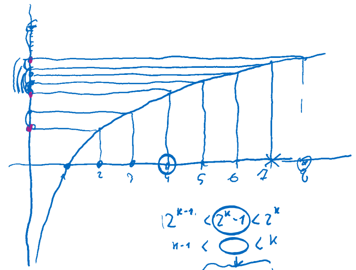
$a_1, a_2, a_4, a_8, a_{16}$

$\boxed{a_{n_k}\to 0}$

$a_{2^k} = \log_2 2^k - \lfloor\log_2 2^k\rfloor = k - \lfloor k\rfloor = 0$

$\boxed{\liminf_{n\to+\infty} a_n = 0}$

$a_{2^k-1}\to 1$?



$2^{k-1} < 2^k-1 < 2^k$

$k-1 < \quad < k$

$a_{2^k-1} = \log_2(2^k-1) - \lfloor\log_2(2^k-1)\rfloor =$

$= \log_2(2^k-1) - (k-1) =$

$= \log_2(2^k-1) - k + 1 =$

$= \log_2(2^k-1) - \log_2 2^k + 1 =$

$= \log_2\left(\frac{2^k-1}{2^k}\right) + 1 =$

$\underbrace{\log_2\left(1-\frac{1}{2^k}\right)}_{\to 0} + 1 \to 1$

$a_n = \log_2 n - \lfloor\log_2 n\rfloor$