Esercitazioni di Analisi 1 - A.A. 21/22 - Callegari

Lezione 19

19 Gennaio 2022

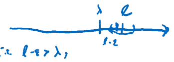
1

146 CAP. 2 MOSTRARE CHE SE (a_n) È LIMITATA ALLORA L'INSIEME $\mathcal{L} = \{ \ell \in \mathbb{R} \mid \ell$ È PUNTO LIMITE DI $(a_n) \}$ È COMPATTO.

DIM ① $\mathcal{L}$ È LIMITATO

SIA $\lambda \in \mathbb{R}$ MAGGIORANTE (MINORANTE) $\{a_n\}_n$, MOSTRIAMO CHE λ È MAGGIORANTE (MINORANTE) PER $\mathcal{L}$.

PER ASSURDO SIA $\ell > \lambda$ T.C. ℓ È PT. LIMITE DI a_n



PRESO $\varepsilon > 0$ T.C. $\ell - \varepsilon > \lambda$,

SICCOME ℓ È P.T. limite $\exists\, a_{n_k} \to \ell$ E QUINDI

$\exists\, a_{n_0} \in (\ell - \varepsilon, \ell + \varepsilon)$. ASSURDO perché

$a_{n_0} > \ell - \varepsilon > \lambda$ ← MAGGIORANTE PER $\{a_n\}$

② $\mathcal{L}$ È CHIUSO.

DATA (ℓ_n) T.C. $\ell_n \to x_0 \Rightarrow \boxed{x_0 \in \mathcal{L}}$

$\forall k \in \mathbb{N}$ Prendo a_{n_k} t.c. $|a_{n_k} - \ell_k| < \frac{1}{k}$.

POSSO FARLO PERCHÉ OGNI ℓ_k È P.T. limite.

SI HA. $\underbrace{\ell_k - \frac{1}{k}}_{\to x_0} < a_{n_k} < \underbrace{\ell_k + \frac{1}{k}}_{\to x_0}$

QUINDI PER T. CONF. $a_{n_k} \to x_0$.

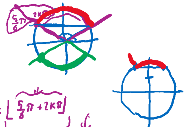
QUINDI $x_0 \in \mathcal{L}$.

2

CALCOLARE LIMINF E LIMSUP DI (a_n) NEI SEGUENTI CASI:

137 CAP. 2 $a_n = \sqrt{n} \sin n$ 137 CAP. 2 $b_n = \sqrt{n} (\sin \sqrt{n})^2$

$$\boxed{\begin{array}{l} \limsup_{n \to +\infty} a_n = +\infty \\ \liminf_{n \to +\infty} a_n = -\infty \end{array}}$$



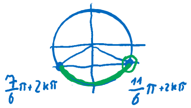
$\forall k \in \mathbb{N}$ $\boxed{n_k} = \lfloor \frac{5}{6}\pi + 2k\pi \rfloor$

$\boxed{\frac{\pi}{6} + 2k\pi} < -1 + \frac{5}{6}\pi + 2k\pi \leq \bullet \leq \boxed{\frac{5}{6}\pi + 2k\pi}$

$\frac{5}{6}\pi - 1 > \frac{\pi}{6}$

$\frac{2}{3}\pi > 1$

$\boxed{a_{n_k}} = \sqrt{n_k} \cdot \sin n_k \geq \frac{1}{2}\sqrt{n_k} \to +\infty$



$n_k = \lfloor \frac{11}{6}\pi + 2k\pi \rfloor$ ←

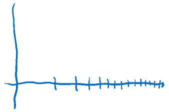
$$\boxed{\sin n_k < -\frac{1}{2}}$$

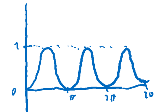
$a_{n_k} = \sqrt{n_k} \cdot \sin n_k \leq -\frac{1}{2}\sqrt{n_k} \to -\infty$

3

$b_n = \sqrt{n} (\sin \sqrt{n})^2$ $b_n = f(\sqrt{n})$

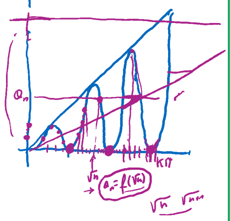
$f(x) = x \sin^2 x$





→ 1) $\limsup_{n \to +\infty} a_n = +\infty$?

→ 2) $\liminf_{n \to +\infty} a_n = 0$



$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \to 0$

n_k t.c.

$\sqrt{n_k} < k\pi < \sqrt{n_k + 1}$

$n_k < \boxed{k^2\pi^2} < n_k + 1$

$n_k = \lfloor k^2 \pi^2 \rfloor$

$0 \leq a_{n_k} = f(\sqrt{n_k}) = \sqrt{n_k} \cdot (\sin \sqrt{n_k})^2 =$

$= \sqrt{n_k} \cdot \sin^2(\sqrt{n_k} - k\pi) =$

$= \sqrt{n_k} \cdot \left(\left| \sin(\sqrt{n_k} - k\pi) \right| \right)^2 =$

$\leq \sqrt{n_k} \cdot \left(\left| \sqrt{n_k} - k\pi \right| \right)^2 \leq$

$\leq \sqrt{n_k} \cdot \left| \sqrt{n_k + 1} - \sqrt{n_k} \right|^2 =$

$= \sqrt{n_k} \cdot \left(\frac{1}{\sqrt{n_k + 1} + \sqrt{n_k}} \right)^2 \leq \sqrt{n_k} \cdot \left(\frac{1}{\sqrt{n_k}} \right)^2 = \frac{1}{\sqrt{n_k}} \to 0$

$|\sin t| \leq |t|$

$n_k = \lfloor k^2\pi^2 \rfloor$

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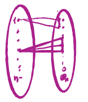
$$\begin{cases} a_n = f(a_{n-1}) \\ a_0 = \lambda \end{cases}$$

$\boxed{\lim_{n \to +\infty} a_n = ?}$

S_n =



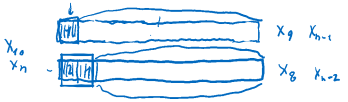
$x_n = ?$

$\begin{cases} x_n = n \cdot x_{n-1} \\ x_1 = 1 \end{cases}$

$x_n = n!$

$x_{n+1} = (n+1)x_n = (n+1) \cdot n! = (n+1)!$

$x_n = \#\{ \varphi \in S_n \mid \forall i = 1, \dots, n \ |\varphi(i) - i| \leq 1 \}$



$x_{10} = x_9 + x_8$

$$\boxed{\begin{cases} x_n = x_{n-1} + x_{n-2} \\ x_1 = 1 \\ x_2 = 2 \end{cases}}$$

$a_n = f(a_{n-1}, a_{n-2})$

$x_n = $?

1 2 3 5 8 13 21 34

$$\boxed{\begin{array}{l} \text{DATO } A_n = \{ \varphi \in S_n \mid \forall i = 1, \dots, n \ |\varphi(i) - i| \leq 1 \} \\ \text{DETTO } x_n = \# A_n \text{ ABBIAMO MOSTRATO CHE} \\ \quad \begin{cases} x_{n+2} = x_{n+1} + x_n \\ x_1 = 1 \\ x_2 = 2 \end{cases} \end{array}}$$

$\boxed{\alpha a_n + \beta b_n}$ $x_n = \dots$

$a_1 = 1$

$a_2 = \frac{1 - \sqrt{5}}{2}$

$\lambda^2 = \lambda + 1$

$\lambda^2 - \lambda - 1 = 0$ $\lambda = \frac{1 \pm \sqrt{5}}{2}$ λ_1, λ_2

→ → $1, \lambda_1, \lambda_1^2, \lambda_1^3, \dots, \lambda_1^n, \dots$ $a_n = \lambda_1^n$

→ → $\dots \lambda_2 \dots$ $b_n = \lambda_2^n$.

$a_{n+2} = a_{n+1} + a_n$??

$\lambda_1^{n+2} = \lambda_1^{n+1} + \lambda_1^n$??

$\lambda_1^2 = \lambda_1 + 1$??

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$\alpha a_{n+2} + \beta b_{n+2} = \alpha a_{n+1} + \beta b_{n+1} + \alpha a_n + \beta b_n$

$\alpha(a_{n+2} - a_{n+1} - a_n) + \beta(b_{n+2} - b_{n+1} - b_n) = 0$?

$x_n = \alpha \lambda_1^n + \beta \lambda_2^n$

$\begin{cases} x_1 = 1 \\ x_2 = 2 \end{cases}$ $\begin{cases} \alpha \lambda_1 + \beta \lambda_2 = 1 \\ \alpha \lambda_1^2 + \beta \lambda_2^2 = 2 \end{cases}$

$\alpha = \dots$

$\beta = \dots$

X