Rome, 20 April 2015 - Rome "Tor Vergata" University

Italian Math Olimpiad Project

5th Math Challenge "Urbi et Orbi"

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1. The *Smiths* (mum, dad and son) are on holiday in *Fattyland*, where public scales are free only for weights $p \geq 100$Kg. If you want to weigh smaller quantities you must pay. Thus, to save money, they weigh themselves in pairs:

a. the weight of mum and dad, together, is exactly 132Kg;
b. the weight of dad and son, together, is exactly 115Kg;
c. mum and son, together, cannot see their weight for free (and they don't want to pay).

Therefore, the whole family jump on the scale. What is the greatest integer value that can appear on the display?

2. A *Satanic* car works as follows: when the number on the odometer has at least 3 consecutive digits equal to **6**, the engine runs in *Diabolic* mode and saves 1 gram of fuel for every kilometer. How many grams of fuel will it save going from 0 to 100000 kilometers.

3. In the words **numeriRUMENI** and **ruminIMUReNE** any small letter appears before its corresponding capital letter. Let N be the number of all their anagrams with that property. Find the last 4 digits of N?

4. Find the least integer $n > 2015$ such that there exists a non-constant polynomial $p(x)$ which satisfies $p(p(p(p(x)))) = (p(x^n))^n$.

5. In my class there are 21 students and I have 10 lemon candies and a lot of apple candies (more than 21). Let N be the number of different ways I can give one candy to each student of my class. What are the last 4 digits of N?

6. A regular tetrahedron $\mathcal{T}$ has volume 7m^3. The total surface of a regular octahedron $\mathcal{H}$ is 72 times the one of $\mathcal{T}$. Find the volume of $\mathcal{H}$ (in m^3).

7. If we expand $(x^7 + x^3 + 1)^{1000}$ and we collect similar terms, we get a polynomial $p(x)$. How many terms has $p(x)$?

8. How many are the integers n, with $1 \leq n \leq 2015$, whose binary form is palindrome.

9. A spherical object hangs in mid-air and is divided into many parts by 48 planes. Every plane is either horizontal or vertical. Find the maximum number of parts that can be obtained.

10. A cricket jumps on the real line. Jumps have length 1. The direction of a jump from a point which is a multiple of 6, is always positive. Otherwise it can be either positive or negative. Find the number of possible sequences of 15 jumps, starting at $x = 2015$.

11. 22 men, each of different age, sit in a circle in such a way that at least 2 of them are older than both of their neighbours. Let N be the number of different ways, up to rotations, in which they can do it. Find the last 4 digits of N.

12. The cube $ABCDEFGH$ has been divided into 8 parallelepipeds by 3 planes (see pictures below). The volumes of the 2 parallelepipeds highlighted in fig. **1** are 36m^3 and 100m^3. The volumes of those in fig. **2** are 225m^3 and 900m^3. Find the volume of the cube (in m^3).

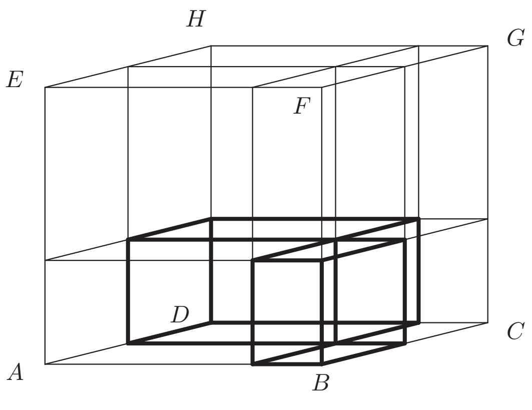

Figure 1

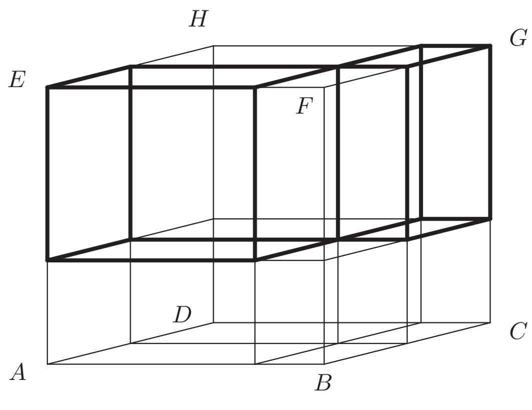

Figure 2

13. Find N_4, where N_0, N_1, N_2, ..., N_{10} are integers such that the following identity holds:

$$x^{20} + y^{20} = N_0(x+y)^{20} + N_1 xy(x+y)^{18} + N_2 x^2y^2(x+y)^{16} + \ldots + N_9 x^9y^9(x+y)^2 + N_{10}x^{10}y^{10}.$$

14. Claudia and Luca are playing a game. Starting with a positive reduced fraction, and taking turns, they do their move, which consists in doing one of the following 2 actions:

- **a.** subtract 1 from the numerator, if it is greater than 1, and then simplify the resulting fraction;
- **b.** subtract 1 from the denominator, if it is greater than 1, and then simplify the resulting fraction.

When a player has to operate on the fraction $1/1$, he cannot move any more and, therefore, he loses the game. Find the number of reduced fractions p/q, with $1 \le p < q \le 9$, such that, when Claudia starts with that fraction, she will win, whatever strategy Luca may choose.

15. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be 2 balls with the same radius and externally tangent. Let $\mathbf{s}_1$, $\mathbf{s}_2$, ..., $\mathbf{s}_n$ be n small balls, all with the same radius and all tangent both to $\mathcal{S}_1$ and to $\mathcal{S}_2$. Moreover, we know that:

- **a.** for any $i = 1, 2, \ldots, n-1$, the balls $\mathbf{s}_i$ and $\mathbf{s}_{i+1}$ are externally tangent;
- **b.** $\mathbf{s}_n$ is externally tangent to $\mathbf{s}_1$.

Find the smallest n such that there exists a large sphere $\mathcal{S}$, containing all the small balls $\mathbf{s}_1$, $\mathbf{s}_2$, ..., $\mathbf{s}_n$ and also $\mathcal{S}_1$, $\mathcal{S}_2$, with $\mathcal{S}$ tangent to $\mathcal{S}_1$ and $\mathcal{S}_2$.

16. For any pair (m, n), where m and n are non-negative integers, define $f(m, n)$ by the following rules:

- **a.** $f(0, k) = f(k, 0) = (k+1)^2$, for any non-negative integer k;
- **b.** $f(m+1, n+1) = f(m, n+1) + f(m+1, n)$, for all non-negative integers n and m.

Find the last 4 digits of $f(10, 10)$.

17. Let $\mathcal{C}$ be a cube with volume 8208m^3 and let r be a straight line through the center of $\mathcal{C}$ and one of its vertices. Let $\mathcal{K}$ be the cube which is symmetric to $\mathcal{C}$ with respect to r. Find the volume of $\mathcal{C} \cap \mathcal{K}$ (in m^3).

18. The triangle ABC is divided into 7 parts by the segments AM, BN e CO (see fig. **3**). The areas of 4 of them are known. Their values (in m^2) are given in the picture. What is the area of the whole triangle ABC (in m^2)?

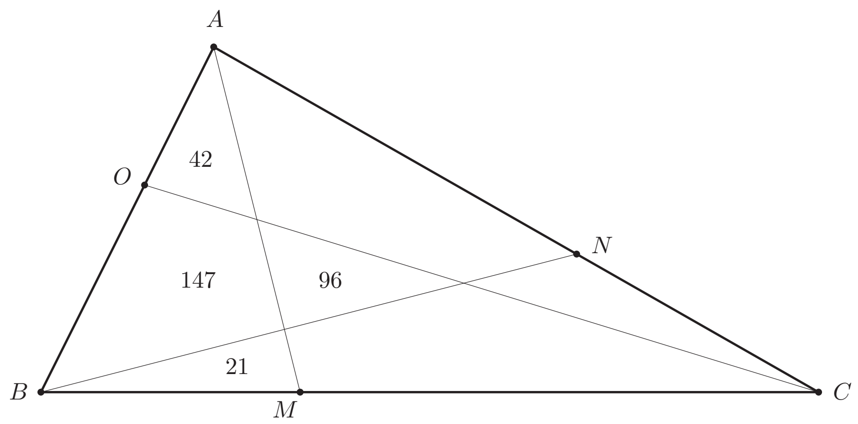

Figure 3

19. A **polyomino** is a plane connected figure, formed by joining a finite number of unit squares edge to edge. Among the polyominoes of area 4, those with largest perimeter, up to translations, rotations and reflections, are the following:

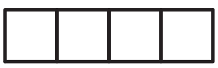

Figure 4

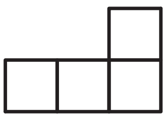

Figure 5

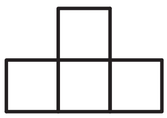

Figure 6

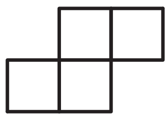

Figure 7

How many polyominoes of area 6, up to translations, rotations and reflections, have the largest perimeter?

20. A cricket is jumping among the cases of a 3×22 chessboard, i.e. a grid with 3 rows and 22 columns. Jumps are allowed only between adjacent squares, i.e. squares with a common side. How many different sequences of jumps start at the first square of the first row and end at the $8th$ square of the third row, hitting every other square exactly once?

(when the game is over)

Useful informations

- `Results and Some Video-Solutions.` They will be published in the night of the **20th of April 2015** at the page:

  http://www.problemisvolti.it/DisfidaMatematicaUrbiEtOrbi.html

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Answers

		answer
Problem 1	:	173
Problem 2	:	280
Problem 3	:	4400
Problem 4	:	2197
Problem 5	:	8576
Problem 6	:	6048
Problem 7	:	6986
Problem 8	:	93
Problem 9	:	8993
Problem 10	:	9232
Problem 11	:	1424
Problem 12	:	2431
Problem 13	:	2275
Problem 14	:	18
Problem 15	:	6
Problem 16	:	6396
Problem 17	:	6156
Problem 18	:	630
Problem 19	:	27
Problem 20	:	6144