Rome, 18th April 2016 - University of Rome "Tor Vergata"

Italian National Math Olimpiad

VI Math Challenge "Urbi et Orbi"

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1. Let $n = 9006000$. Find the smallest integer $d > \sqrt{n}$ such that d divides n.

2. Let $p(x)$ be a polynomial such that the straight line $y = x$ intersects the graph of $y = p(x)$ exactly in 17 points. What is the minimum possible degree of $p(x)$?

3. The barrel **A** contains 2016 litres of wine and the barrel **B** contains 2016 litres of water. Take n litres of wine (n is positive integer) from **A** and pour them into **B** then, after mixing accurately, take the same number n of litres of the liquid of **B** and pour them into **A**.
Find the minimum value of n such that, at the end, at least one third of the liquid of **A** is water.

4. A stick, whose length is 2016 mm, is divided into 63 smaller pieces, whose lengths are in arithmetic progression and such that the larger piece is 63 times the smaller one. How long is (in millimeter) the larger piece?

5. Find the least positive integer n such that $n!$ has exactly 1000 final zeros. (If it doesn't exist, give 9999 as an answer)

6. A roulette player believes that, in order to have better chance of victory, he must choose 3 distinct numbers such that their product is a multiple of 8. How many such choices can he make? (Classic roulette has 37 numbers: 0, 1, 2, ... 36.)

7. On the *Nowhere* island there is an epidemic of *Sincerity Desease*. It is a very dangerous desease: sick people seem normal but they are not because they are no more able to lie. In order to identify sick people, the board of health has developed a very reliable medical test: it gives the right answer in the 99% of cases, both with sick people and with healthy ones.
After the screening of the whole population (1 million), the citizens whose test result is positive are 10147.
How many citizens are really sick? (if you think they are more than 9999, give 9999 as an answer)

8. $ABCD$ is a trapezoid with bases $AB = 89$ and $CD = 14$. Moreover $AD = 199$ and there is a point P on AB such that, if $Q = PC \cap DB$, the areas of APQ and CQB are both equal to 1050. Find the area of $ABCD$.

9. How many different tessellations of a 10×1-rectangle (see fig. **1**) are possible using black or white 1×1-squares and grey 2×1-rectangles? (Tessellations that do not use all tree types of tiles are also allowed)

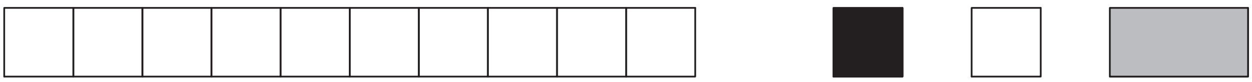

fig. 1

10. Let $A \equiv (1800, 1440)$, $B \equiv (720, 450)$ and $C \equiv (3600, 540)$ be three points in the cartesian plane Oxy. Let $P \equiv (x, y)$ be a point inside the triangle ABC such that the ratios between the areas of the triangles PBC, PCA e PAB, in this order, are $9 : 1 : 5$. Find $x + y$.

11. Claudia and Luke are playing with the following rules:

(**a**) at the beginning a stick of integer length is placed on the table;

(**b**) alternately, each player cuts the stick into two pieces of positive integer lengths and such that the difference of the two lengths is 0 or 1, then he/she puts one piece on the table and throws the other into the fire;

(**c**) the loser is the one that, on his turn, find on the table a stick of length 1.

At a certain point, when it is Claudia's turn, the length of the stick is a number n of 4 digits and Luke realizes that, no matter how Claudia will move, he will win for sure.
Find the maximum value of n.

12. Let $ABCDE$ be a pentagon on the cartesian plane. Let $P \equiv (-21, 26)$, $Q \equiv (-42, -65)$, $R \equiv (7, -52)$, $S \equiv (49, 13)$ and $T \equiv (35, 78)$ be the midpoints of AB, BC, CD, DE ed EA, respectively. Find the sum of the coordinates of A.

13. In fig. **2** the shorter side of the rectangle is equal to 66 and the radii of the three circles are all equal to 15. Find the sum of the areas of the two grey triangles.

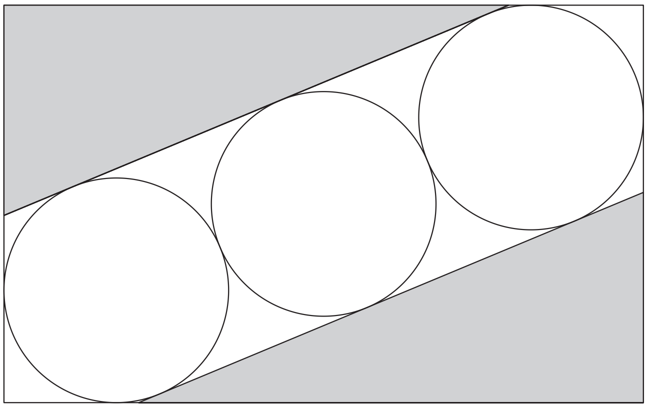

fig. 2

14. A regular polygon $\mathcal{P}$ with 9797 sides has, obviously, 9797 symmetry axes. Let us color every side of $\mathcal{P}$ with black or white (not all sides with the same color). How many are, at most, the symmetry axes of the polygon that are symmetry axes also for the coloring.

15. Find the smallest positive integer n such that both $529n + 1$ and $528n + 1$ are perfect squares.

16. Claudia has 102 Lego bricks, each one labelled with a different integer between 1 and 102. She wants to use all the bricks to build a block as in fig. **3**, where we can see blocks made of 3, 6 and 24 bricks, respectively. Moreover Claudia would like also that, whenever a brick **A** lies on a brick **B**, then **A** must be labelled by a number greater than **B**. How many different blocks of 102 bricks can be made by Claudia?

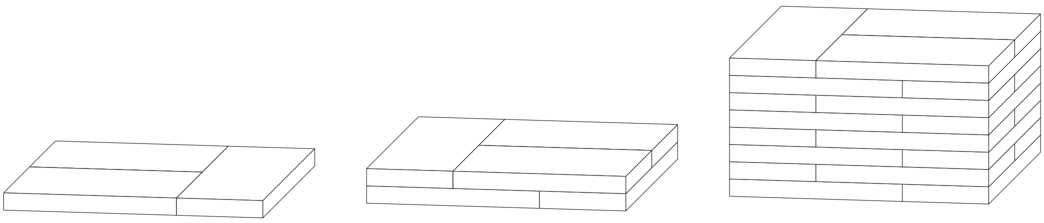

fig. 3

17. Find how many permutations $(a_1, a_2, \dots, a_{13})$ of the integers between 1 and 13 are such that $a_i < a_{2i}$ and $a_i < a_{2i+1}$ for any $i = 1, \dots, 6$. (If the result is greater than 9999 give, as an answer, the 4 less-significant digits)

18. A cricket jumps on the cartesian plane. A jump will be considered **good** if, starting at point (x, y), it ends in a point of the form $(x + s, y)$ or $(x, y + s)$, with $s \in \{1, 3, 5, 11, 23, 47, 91, 2016\}$. How many are the sequences of 8 consecutive **good** jumps of different lengths, such that the first jump starts at $(0, 0)$ and the other jumps never overstep the line $y = x$? (If the result is greater than 9999 give, as an answer, the 4 less-significant digits)

19. The triangle ABC has $AB = 52$, $BC = 60$ and $CA = 56$. Let P be a point of BC such that the incircles of ABP and ACP have the same radii. Find the square of the length of AP.

20. A cube has vertices $ABCDEFGH$ and volume 10125. Let $\mathcal{P}$ be the pyramid whose basis is the face $ABCD$ of the cube and with vertex in the center of $EFGH$. Similarly, let $\mathcal{P}$ be the pyramid (equal to $\mathcal{Q}$) with basis $ABFE$ and vertex in the center of $DCGH$. Find the volume of $\mathcal{P} \cap \mathcal{Q}$.

(when the game is over)

Usefull informations

- Results and some Video-Solutions: They will be pubblished on **April 18, 2016** at the website **www.problemisvolti.it**.
- Call for Problems/Help: Do you want to help us with problems/solutions/translations? Please, contact Emanuele Callegari: callegar@mat.uniroma2.it

Solutions

	answer
Problem 1 :	3002
Problem 2 :	17
Problem 3 :	1008
Problem 4 :	63
Problem 5 :	4005
Problem 6 :	4164
Problem 7 :	150
Problem 8 :	9167
Problem 9 :	5741
Problem 10 :	3402
Problem 11 :	6826
Problem 12 :	118
Problem 13 :	2523
Problem 14 :	101
Problem 15 :	8456
Problem 16 :	6128
Problem 17 :	6880
Problem 18 :	4050
Problem 19 :	2016
Problem 20 :	1380