Italian National Math Olimpiad

VII Math Challenge "Urbi et Orbi"

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1. The base-3 representation of the positive integer n has 400 digits with a repeating group of four digits, in the following way:

$$n = \overbrace{\overbrace{2011}\,\overbrace{2011}\,\overbrace{2011}\ldots\overbrace{2011}}^{100}$$

Find the last 4 digits of its base-9 representation.

2. One of the faces of a rectangular parallelepiped has area 40, another face has area 105 and a third face has area 168. Find the volume of the parallelepiped.

3. The *Neverending Land* is a large flatland, where a Cartesian coordinate system is fixed with origin at the Maths Department. The roads are all the lines that are parallel to one of the axes and whose distance from it is an integer number of kilometers.
Given any two points P and Q on the road network, we define the **road distance** between P and Q as the length of the shortest road path between P and Q.
Luke knows that Claudia's house has a **road distance** of 17.5 Km from the Maths Department and a **road distance** of 16.5 Km from the Economy Department, which is placed on the point of coordinates $(15, 16)$.
Find the number of points of the road network where Claudia's house could be.
(if their number is not finite, give, as an answer, 9999)

4. Find the number of acute-angled isosceles triangles with integer sides and perimeter 2017.

5. The positive integers m, n and k satisfy the conditions $\gcd(m, n, k) = 1122332211^2$ and $\text{lcm}(m, n, k) = 1122332211^3$. How many possible values are there for the product $m \cdot n \cdot k$?

6. The triangle ABC, right-angled in B, has $AB = 60$ and $BC = 80$. Another triangle, equal to ABC, is placed on the same plane of ABC, in such a way that the vertex of the right angle is placed at A and the hypotenuse is parallel to AB. Moreover we know that the intersection of the two triangles has strictly positive area.
Find the value of this area.

7. How many right-angled triangles with integer sides are there whose inscribed circle has diameter 600?

8. The lengths of the three heights of a triangle are 840 mm, 728 mm e 780 mm. Find the area of the triangle in cm^2.

9. Find the least positive integer n such that $\sqrt{n(n + 2017)}$ is an integer. (If the result is greater than 9999 give, as an answer, the 4 less significant digits)

10. Let ABC be an isosceles triangle with base BC. Let D be the intersection between the side AC and the bisector of $\widehat{ABC}$. Similarly let E be the intersection between the side AB and the bisector of $\widehat{ACB}$. Moreover let the circumcenter O be on the segment DE and let the radius of the circumcircle be 42.
Find the square of the height relative to BC.

11. On the *Nowhere* island candy-machines have two buttons: if we push the first one you obtain a fixed number x of candies, while if we push the second one you obtain another fixed number y of candies. Leo and his 10 friends push 3 times the first button and 7 times the second one. They obtain a certain number of candies that they try to divide into 11 equal parts, but they can't because there is a remainder of 9. Other 2 friends arrive and push twice both buttons: they obtain some more candies that they add to the others. Then they try to divide all candies into 13 equal parts but they still can't because they have a remainder of 2.
Find all the possible pairs (x, y) that satisfy also the conditions $0 < x \leq 75$ and $0 < y \leq 100$.
Give as an answer the number obtained as sum of all the possible values of y plus the double of the sum of all the possible values of x.

12. Johnny's garage is a large room, with 22 columns, no windows and one door. The inner surface has been completely tiled (walls, columns, floor and ceiling). Tiles are irregular polygones with an arbitrary number of sides. Any two tiles have in common a side, a vertex or nothing. Let S be the total number of sides and V the total number of vertex (a side belonging to two tiles or a vertex belonging to more than one tiles must be counted once). If $S - V = 759$, how many tiles have been used?

13. Let $p_1, p_2, p_3, \ldots p_{10}$ be distinct prime numbers and let $n = (p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_{10})^{2017}$.
Moreover, let $\mathcal{D}$ denote the set of all positive divisors of n, 1 and n included, and let $\mathcal{J}$ be any subset of $\mathcal{D}$ such that the product of any two distinct elements of $\mathcal{J}$ is never a perfect square.
What is the maximum number of elements of $\mathcal{J}$?

14. Find all the triplets (p, q, n), with p and q prime numbers and n positive integer, such that

$$p(p+1) + q(q+1) = n(n+1)$$

If (p_1, q_1, n_1), (p_2, q_2, n_2), $\ldots$, (p_k, q_k, n_k) are all the possible triplets, give as an answer $p_1 \cdot q_1 \cdot n_1 + p_2 \cdot q_2 \cdot n_2 + \ldots + p_k \cdot q_k \cdot n_k$. (if there are no solutions give 0 as an answer; if the solutions are infinite give 9999 as an answer; if the required sum is greater then 9999 give as an answer the 4 less significant digits of the result)

15. A convex quadrilater $ABCD$ is divided into two pieces of area 432 and 576 by each diagonal. Let $\mathcal{S}$ be the locus of the points P of the plane such that the sum of the areas of APB, BPC, CPD and DPA does not exceed three times the area of $ABCD$. What is the area of $\mathcal{S}$? (If the result is greater than 9999 give as an answer the 4 less significant digits)

16. Let $A = \{1, 2, 3, \ldots, 10\}$. How many subsets of A contain exactly two elements which are consecutive numbers?

17. An anagram of **AAABBBAAABBB** is said to be **crazy** if, when cut into two pieces, in each piece the difference between the number of letters **A** and the number of letters **B** is never greater than 2.
How many **crazy** anagrams are there?

18. A triplet (A, B, C), where A, B and C are subsets of $\{1, 2, 3, 4, 5\}$, will be said to be **good** if $A \cap B \cap C$ is empty but $A \cap B$, $B \cap C$, and $C \cap A$ are not. How many triplets are **good**? (remember that a *triplet* is ordered: triplets with the same elements but in different orders must be considered different)

19. Let $\mathcal{F}$ be the family of all pairs of two non negative integers that are not both equal to zero. In how many ways can we obtain $(2, 2017)$ as the sum of one or more elements of $\mathcal{F}$? Two sums with the same terms but in different order should be considered distinct. (Recall that the sum of two pairs (a, b) and (c, d) is the pair $(a + c, b + d)$. If the result is greater than 9999 give as an answer the 4 less significant digits).

20. The surface $\mathcal{S}$ is generated by rotating the hyperbola $y^2 = x^2 + 1$ around the x-axis. Let r, s, t and ℓ, be 4 lines, no two of which belongs to the same plane, but such that r, s, t are contained in $\mathcal{S}$ while ℓ is not.
How many lines, at most, intersect all the 4 lines r, s, t, ℓ?

(when the game is over)

Useful informations

- `Results and some Video-Solutions:` They will be published on **April 10, 2017** at the web-site **www.problemisvolti.it**.
- `Call for Problems/Help:` Do you want to help us with problems/solutions/translations? Please, contact Emanuele Callegari: `callegar@mat.uniroma2.it`

Answers to the problems

Problem 1	:	6464
Problem 2	:	840
Problem 3	:	5
Problem 4	:	418
Problem 5	:	162
Problem 6	:	666
Problem 7	:	45
Problem 8	:	3549
Problem 9	:	6064
Problem 10	:	5292
Problem 11	:	7462
Problem 12	:	716
Problem 13	:	1024
Problem 14	:	192
Problem 15	:	6961
Problem 16	:	235
Problem 17	:	486
Problem 18	:	1830
Problem 19	:	7488
Problem 20	:	2