Florence, 14 April 2018

European Girls' Mathematical Olympiad

Mathematics Challenge for Teams

Problems by: F.Ballini, E.Callegari, L.Ferrigno, G.Marini, P.Perfetti, A.Sgueglia, R.Tauraso, R.Vacca, A.Veronese.

1. How many integers n, with $9500 \leq n \leq 9700$, are relatively prime with 9595?

2. Let p and q be prime numbers such that $pq = 11663$. Find $p + q$.

3. Consider the list of all positive divisors of 720^2, in ascending order from 1 to 720^2. Find the $68th$ number of the list.

4. Consider the list, in ascending order, of all positive integers that are either powers of 7 or sum of mutually distinct powers of 7. Thus the first elements of the list are: 1, 7, 8, 49, 50, 56, ... Find the last 4 digits of the $49th$ number of the list.

5. Find the smallest positive integer such that the product of all its positive divisors is 72^{35}.

6. The area of the triangle $\mathcal{T}$ is a 4 digits integer. Moreover, $\mathcal{T}$ has two altitudes of length 99 and 101. How many different values are possible for the area of $\mathcal{T}$?

7. In the triangle ABC we have $AB = 50$ and $BC = AC = 2018$. Let P and Q be the points of the side AC such that the incircle is tangent in P and the excircle is tangent in Q. Find the length of PQ.

8. A number x is the sum of mutually distinct $2017th$-powers of odd numbers. Let r be the remainder of the division of x by 2018. How many different values are possible for r?

9. In the *Fox Game* there are two bags full of coins: the First One with 7395 coins and the Second One with 6503 coins. Two types of moves are allowed:

move of type 1 : remove x coins from the First Bag, with $1 \leq x \leq 18$,

move of type 2 : remove y coins from the Second Bag, with $1 \leq y \leq 30$,

Two players make alternately their moves. The loser is the one who removes the last coin. A move that guarantees the player the existence of a winning strategy is called *Fox Move* and its corresponding *Fox Number* is $100m + t$ where m is the type of move (1 or 2) and t is the number of removed coins.
Find the sum of all the *Fox Numbers* of the initial position of the game. (If the initial position has no *Fox Moves* then the answer is 0).

10. In the triangle ABC the altitudes from the points A, B and C have length 63, 28 and 27, respectively. Let P be a point inside the triangle, whose distances from the sides AB and AC are 6 and 4, respectively. Find the distance from P to BC.

11. Find the number of ordered pairs (x, y) of positive integers such that:

$$xy + 3 \cdot \mathbf{LCM}(x, y) = 2018 + 6 \cdot \mathbf{GCD}(x, y),$$

where **LCM** means "least common multiple" and **GCD** means "greatest common divisor".

12. Mark and Claudia are playing the following game: they put in a box 673 white balls and 672 black balls, then they pick up one ball: if it is white the winner is Claudia, otherwise the winner is Mark. But the game is not fair, so Mark proposes to change the rules in the following way: first they randomly remove 5 balls (without looking at them) and only then they pick up the ball to decide the winner. With the new rules, what is the probability that Claudia is the winner? (Write the solution in the form $\frac{m}{n}$, with m and n positive and relatively prime, and give $m + n$ as answer).

13. Let P be a polynomial with integer coefficients such that $P(1) = 11$ and $P(12) = k$, where k is a positive 4 digits integer. How many different values are possible for k?

14. Find the largest integer n such that the inequality $x^{7x} \leq x^n + 1 - x$ holds for every $0 < x \leq 1$.

15. Let $\mathcal{F}$ be the family of all 4-tuples (a,b,c,d) of integers, with $0 \le a,b,c,d \le 100$, such that $\max\{a,b\} \le \max\{c,d\}$ and $\max\{a,c\} \le \max\{b,d\}$. Find the number of elements of $\mathcal{F}$. (If the result has more than 4 digits, give only the last 4 digits as answer.)

16. The area of the triangle ABC is 5040 and $AC > AB$. Let AM and AN be, respectively, the bisector and the median from vertex A. The area of AMN is 840 and $AM = 70$. Find the length of the median from the vertex B.

17. P is a polynomial of degree 7 such that $(x+1)^4$ divides $P(x) - 32$ and $(x-1)^4$ divides $P(x) + 32$. Find the value of $P(2)$.

18. A 6-tuple $(n_1,\, n_2,\, n_3,\, n_4,\, n_5,\, n_6)$ of positive integers is called **hexagonal** if there exists a hexagon, with all internal angles equal to 120°, such that, clockwise, the lengths of the sides are n_1, n_2, n_3, n_4, n_5, n_6, respectively. How many hexagonal 6-tuples are there such that the perimeter of the corresponding hexagon is 15?

19. All the 12 pentagonal faces of a soccer ball (i.e. a truncated icosahedron) are black while the 20 hexagonal faces are blue or red. If there are no blue hexagons that share an edge, what is the maximum number of blue faces?

20. The rectangle 2×10 in picture **1** is covered exactly and without overlapping by 5 rectangles with integer sides. Picture **2** shows an example of admissible covering. How many different coverings are possible?

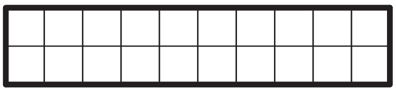

fig. 1

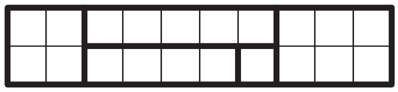

fig. 2

21. Let $ABCD$ be a cyclic quadrilateral whose diagonals intersects in M and are orthogonal. Let P, Q, R, and S be the projections of M onto DA, AB, BC, e CD, respectively. Finally let $AB = 51$, $CD = 68$ and $DA = 75$. Find the area of $PQRS$. (Write the solution in the form $\frac{m}{n}$, with m and n positive and relatively prime, and give the last 4 digits of $m+n$ as answer).

22. Let $\mathcal{S}$ be the set of positive integers whose ternary representation (i.e. base 3) has at most 2018 digits, each of them equal to '1' or to '0'. Find the smallest integer k which is larger than the sum:

$$720 \cdot \sum_{n \in \mathcal{S}} \frac{3n+2}{n(3n+1)}.$$

23. The lengths of the sides of the triangle ABC are 3, 5 and 7. Let G be the centroid of ABC and let DEF be the triangle whose vertices are the projections of G onto the sides of ABC. Find the ratio $\mathcal{R}$ of the area of DEF to the area of ABC. (Write the solution in the form $\frac{m}{n}$, with m and n positive and relatively prime, and give $m+n$ as answer).

24. In a scalene triangle the lengths a, b and c of the sides are integers such that $a^2 + b^2 + c^2 = 2018$. Find the perimeter of the triangle.

(when the game is over)

Useful informations

- Results and some Video-Solutions: They will be published on **April 16, 2018** at the website **www.problemisvolti.it**.

- Call for Problems/Help: Do you want to help us with problems/solutions/translations? Would you like to partecipate on line to the 2019 team competition? Please, contact Emanuele Callegari: callegar@mat.uniroma2.it

Answers to the problems

Problem 1	:	151
Problem 2	:	216
Problem 3	:	720
Problem 4	:	9209
Problem 5	:	5184
Problem 6	:	5000
Problem 7	:	1968
Problem 8	:	2018
Problem 9	:	220
Problem 10	:	40
Problem 11	:	12
Problem 12	:	2018
Problem 13	:	819
Problem 14	:	8
Problem 15	:	6201
Problem 16	:	96
Problem 17	:	356
Problem 18	:	22
Problem 19	:	8
Problem 20	:	2562
Problem 21	:	4229
Problem 22	:	2160
Problem 23	:	671
Problem 24	:	76