Analisi Matematica 1 - Lezione 8

15 Ottobre 2014 - docente: Prof. **Emanuele Callegari** - Università di Roma ***Tor Vergata***

Teorema $\forall a>1$ e $\forall \alpha>0$ le successioni

$\log_a n$, n^α, a^n, $n!$ e n^n

tendono a $+\infty$, per $n\to+\infty$.

Inoltre ① $\lim_{n\to+\infty} \frac{n!}{n^n}=0$ ② $\lim_{n\to+\infty} \frac{a^n}{n!}=0$

③ $\lim_{n\to+\infty} \frac{n^\alpha}{a^n}=0$ ④ $\lim_{n\to+\infty} \frac{\log_a n}{n^\alpha}=0$

Dim

$\lim_{n\to+\infty} n^n=+\infty$ $\qquad n^n>n\to+\infty$ (confronto)

$\lim_{n\to+\infty} n!=+\infty$ $\qquad n!=1\cdot2\cdot3\cdots n>n\to+\infty$ (confronto)

$\lim_{n\to+\infty} a^n \boxed{=+\infty}$

$$a^n=(1+\delta)^n\geq \underbrace{1+n\delta}_{\to+\infty}$$

$a^n \to +\infty$, $\delta > 0$

MOSTRIAMO PRIMA DIS. DI BERNOULLI

$(1+a)^n\geq 1+na$ $\quad$ (con $a>0$, $n\in\mathbb{N}$)

Per induzione su n

1) $n=0$ vera SI

2) $n=k \Rightarrow n=k+1$

$(1+a)^{k+1}=(1+a)^1\cdot(1+a)^k\geq(1+a)(1+ka)$
$=1+(k+1)a+ka^2\geq 1+(k+1)a$

$$\boxed{\lim_{n\to+\infty} \log_a n = +\infty}$$

$\forall M \in \mathbb{R}$ prendo a^M e vero che def. in n

$$n > a^M \iff \boxed{\log_a n} > \log_a a^M = \boxed{M}$$

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① $\displaystyle\lim_{n\to+\infty} \frac{n!}{n^n} = 0$ Basta mostrare che $\displaystyle\lim_{n\to+\infty} \frac{n^n}{n!} = +\infty$

$$\underbrace{\boxed{\frac{n^n}{n!}}}_{\to +\infty} = \frac{\overbrace{n\cdot n\cdot n\cdots n}^{n}}{1\cdot 2\cdot 3\cdots n} = \frac{n}{1}\cdot\frac{\overbrace{n\cdot n\cdots n}^{n-1}}{\underbrace{2\cdot 3\cdots n}_{n-1}} > \underbrace{n}_{\to +\infty}$$

(confronto)

② $\displaystyle\lim_{n\to+\infty} \frac{a^n}{n!} = 0$ Basta mostrare che $\displaystyle\lim_{n\to+\infty} \frac{n!}{a^n} = +\infty$

def. in n $(n \geq \lfloor a \rfloor + 1)$

$$\underbrace{\frac{n!}{a^n}}_{\to +\infty} = \frac{1\cdot 2\cdots \lfloor a\rfloor}{\underbrace{a\cdot a\cdots a}_{\lfloor a\rfloor}}\cdot\frac{\overbrace{(\lfloor a\rfloor+1)\cdots(n-1)}}{\underbrace{a\cdot a\cdots a}_{n-(\lfloor a\rfloor+1)}}\cdot\frac{n}{a} \geq \frac{(\lfloor a\rfloor)!}{a^{\lfloor a\rfloor}}\cdot 1\cdot\frac{n}{a} = \underbrace{\boxed{\frac{(\lfloor a\rfloor)!}{a^{\lfloor a\rfloor+1}}}}_{\text{costante}}\cdot \underbrace{n}_{\to +\infty}$$

(confronto)

③ $\lim_{n\to+\infty} \frac{n^\alpha}{a^n} = 0$ ovvero che $\lim_{n\to+\infty} \frac{a^n}{n^\alpha} = +\infty$

$\boxed{\text{I° passo}}$ Mostriamo che vale per $\alpha = \frac{1}{2}$

Devo mostrare che $\lim_{n\to+\infty} \frac{a^n}{\sqrt{n}} = +\infty$

Si ha:

$$\boxed{\frac{a^n}{\sqrt{n}}} = \frac{(1+\overset{>0}{\delta})^n}{\sqrt{n}} \geq \frac{1+n\delta}{\sqrt{n}} > \frac{n\delta}{\sqrt{n}} = \boxed{\delta\cdot\sqrt{n}}$$

$\boxed{\frac{a^n}{\sqrt{n}}} \to +\infty$ (confronto) $\boxed{\delta\cdot\sqrt{n}} \to +\infty$

$\boxed{\text{II° passo}}$ $\alpha > 0$ qualsiasi

$$\boxed{\frac{a^n}{n^\alpha}} \begin{cases} \overset{\alpha<\frac{1}{2}}{>} \frac{a^n}{\sqrt{n}} \\ \overset{\alpha>\frac{1}{2}}{=} \left(\frac{\left(a^{\frac{1}{2\alpha}}\right)^n}{\sqrt{n}}\right)^{2\alpha} = \left(\frac{\overset{\boxed{b>1}}{b^n}}{\sqrt{n}}\right)^{2\alpha} \overset{\text{def. } n}{>} \boxed{\frac{b^n}{\sqrt{n}}} \end{cases}$$

$\boxed{\frac{a^n}{n^\alpha}} \to +\infty$ (confronto) $\boxed{\frac{b^n}{\sqrt{n}}} \to +\infty$

$$\lim_{n\to+\infty} \frac{\log_a n}{n^\alpha} = 0 \iff \lim_{n\to+\infty} \frac{n^\alpha}{\log_a n} = +\infty$$

$$\boxed{\frac{n^\alpha}{\log_a n}} = \frac{a^{\log_a(n^\alpha)}}{\log_a n} = \frac{(a^\alpha)^{\log_a n}}{\log_a n} = \frac{(b)^{\log_a n}}{\log_a n} >$$

$\boxed{\frac{n^\alpha}{\log_a n}} \to +\infty$

$$> \frac{b^{\lfloor \log_a n \rfloor}}{\lfloor \log_a n \rfloor + 1} = \frac{1}{b} \cdot \underbrace{\frac{b^{\lfloor \log_a n \rfloor + 1}}{\lfloor \log_a n \rfloor + 1}}_{\to +\infty}$$

(conforts)