Analisi Matematica 1 – Lezione 9 (I parte)

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IL NUMERO "e"

Teorema 1 Date $a_n=\left(1+\frac{1}{n}\right)^n$ e $b_n=\left(1+\frac{1}{n}\right)^{n+1}$

Allora
1) a_n è crescente
2) b_n è decrescente
3) entrambe tendono allo stesso limite, compreso tra 2 e 3.

Dim 1) a_n è crescente

$$a_{n+1} \overset{?}{\geq} a_n \qquad \forall n\in \mathbb{N}-\{0\}$$

$$\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^{n} \qquad (?)$$

$$\left(1+\frac{1}{n+1}\right)^{\frac{n+1}{n}} > 1+\frac{1}{n} \qquad (?)$$

$$\left(1+\frac{1}{n+1}\right)^{\frac{n+1}{n}} \underset{\text{Bernoulli}}{>} 1+\frac{\cancel{n+1}}{n}\cdot\frac{1}{\cancel{n+1}} = \boxed{1+\frac{1}{n}}$$

2) b_n è decrescente

$$b_{n+1} < b_n \quad \cancel{?}$$

$$\left(1+\frac{1}{n+1}\right)^{n+2} < \left(1+\frac{1}{n}\right)^{n+1} \quad \cancel{(?)}$$

$$\parallel \qquad\qquad\qquad \parallel$$

$$\left(\frac{n+2}{n+1}\right)^{n+2} \qquad \left(\frac{n+1}{n}\right)^{n+1}$$

$$\left(\frac{n+1}{n+2}\right)^{n+2} > \left(\frac{n}{n+1}\right)^{n+1} \quad \cancel{(?)}$$

$$\left(\frac{n+2-1}{n+2}\right)^{\frac{n+2}{n+1}} > \left(\frac{n+1-1}{n+1}\right) = 1-\frac{1}{n+1} \quad \cancel{(?)}$$

$$\left(1-\frac{1}{n+2}\right)^{\frac{n+2}{n+1}} > 1+\frac{\cancel{n+2}}{n+1}\cdot\left(-\frac{1}{\cancel{n+2}}\right) = 1-\frac{1}{n+1}$$

Bernoulli

$\boxed{\text{OSS. 1}}$ $\quad \forall n \in \mathbb{N}-\{0\}$

$$a_n < b_n \quad ?$$

$$\left(1+\frac{1}{n}\right)^{n} \qquad \left(1+\frac{1}{n}\right)^{n+1}$$

OSS 2

$$\forall\, i,j \in \mathbb{N} - \{0\} \qquad a_i < b_j$$

Caso particolare

$a_{75} < b_{90}$

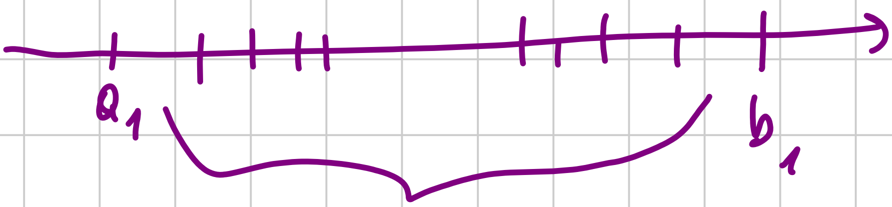
$a_{75} < a_{90} < b_{90}$

Prendo $k = \text{MAX}\{i,j\}$

Avrò che

$$a_i \leq a_k < b_k \leq b_j$$

OSS. 3 Tutti i termini delle 2 successioni sono nell'intervallo $[a_1, b_1]$



OSS 4 Da ① e ② e $\boxed{\text{OSS 3}}$ segue che (a_n) e (b_n) sono monotone e limitate e quindi hanno limite finito.

$$a_n \to l$$

$$b_n \to L$$

③ $l = L$ (?)

$$b_n = \left(1+\frac{1}{n}\right)^{n+1} = \underbrace{\left(1+\frac{1}{n}\right)^{n}}_{a_n} \cdot \left(1+\frac{1}{n}\right)^{1}$$

$$\left.\begin{array}{c} b_n = \underbrace{\underbrace{\left(1+\frac{1}{n}\right)}_{\downarrow \, 1} \cdot \underbrace{a_n}_{\downarrow \, \ell}}_{\downarrow \, \ell} \\ \downarrow \\ L \end{array}\right] \Rightarrow L = \ell$$

Ultima cosa:

$$2 < \ell < 3$$

Sì

$$\ell > a_2 = \left(1+\frac{1}{2}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4} > 2$$

$$b_2 = \left(1+\frac{1}{2}\right)^3 = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$$

$$b_5 = \left(1+\frac{1}{5}\right)^6 = \left(\frac{6}{5}\right)^6 = \dots < 3$$

$\boxed{\text{Def.}}$ Indichiamo con "e" il valore del limite trovato nel T. 1

$\boxed{\text{T. 2}}$ (Gener. di T. 1)

Se $a_n \to +\infty$ allora $\left(1+\frac{1}{a_n}\right)^{a_n} \to e$

$\boxed{\text{Dim}}$

$$\underbrace{\left(1+\frac{1}{\lfloor a_n\rfloor+1}\right)^{\lfloor a_n\rfloor}}_{} < \underbrace{\left(1+\frac{1}{a_n}\right)^{a_n}}_{\to e} < \underbrace{\left(1+\frac{1}{\lfloor a_n\rfloor}\right)^{\lfloor a_n\rfloor+1}}_{\to e}$$

$$\underbrace{\underbrace{\left(1+\frac{1}{\lfloor a_n\rfloor+1}\right)^{\lfloor a_n\rfloor+1}}_{\to e}\cdot\underbrace{\left(1+\frac{1}{\lfloor a_n\rfloor+1}\right)^{-1}}_{\to 1}}_{\to e}$$

$\boxed{\text{ES. 1.7}}$

$$\lim_{n\to+\infty}\left(1+\frac{2}{n}\right)^{n} =$$

$$a_n^{b_n} \to e^L, \quad a_n \to e, \quad b_n \to L$$

$$=\lim_{n\to+\infty}\left(\left(1+\frac{1}{\frac{n}{2}}\right)^{\frac{n}{2}}\right)^{2} =$$

$$=\lim_{n\to+\infty}\underbrace{\left(1+\frac{1}{\frac{n}{2}}\right)^{\frac{n}{2}}}_{\to e}\cdot\underbrace{\left(1+\frac{1}{\frac{n}{2}}\right)^{\frac{n}{2}}}_{\to e} = e^2$$

$\boxed{\text{ES. 11.7}}$ $\lim_{n\to+\infty}\left(1+\frac{1}{n^2}\right)^{2n^2+5} =$

$$= \lim_{n\to+\infty} \underbrace{\left(1+\frac{1}{n^2}\right)^{n^2}}_{\downarrow\, e} \cdot \underbrace{\left(1+\frac{1}{n^2}\right)^{n^2}}_{\downarrow\, e} \cdot \underbrace{\left(1+\frac{1}{n^2}\right)^{5}}_{\downarrow\, 1} = e^2$$

$\boxed{\text{E. 14.7}}$ $\lim_{n\to+\infty}\left(\frac{n+9}{n+5}\right)^{3n} =$

$$= \lim_{n\to+\infty}\left(\frac{n+5+4}{n+5}\right)^{3n} =$$

$$= \lim_{n\to+\infty} \left(\underbrace{\left(1+\frac{1}{\frac{n+5}{4}}\right)^{\frac{n+5}{4}}}_{e} \right)^{\frac{12n}{n+5} \to 12} = e^{12}$$

$$\frac{12n}{n+5} = \frac{12}{1+\frac{5}{n}} \to 12$$