

# Analisi Matematica 1 - Lezione 9 (I parte)

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## IL NUMERO "e"

**Teorema 1** Dato  $a_n = \left(1 + \frac{1}{n}\right)^n$  e  $b_n = \left(1 + \frac{1}{n}\right)^{n+1}$

Allora

1)  $a_n$  è crescente

2)  $b_n$  è decrescente

3) entrambe tendono allo stesso limite, compreso tra 2 e 3.

**Dim**

1)  $a_n$  è crescente

$$a_{n+1} \geq a_n \quad \forall n \in \mathbb{N} - \{0\}$$

$$\left(1 + \frac{1}{n+1}\right)^{n+1} > \left(1 + \frac{1}{n}\right)^n \quad (?)$$

$$\left(1 + \frac{1}{n+1}\right)^{\frac{n+1}{n}} > 1 + \frac{1}{n} \quad (?)$$

$$\left(1 + \frac{1}{n+1}\right)^{\frac{n+1}{n}} > 1 + \frac{\cancel{n+1}}{n} \cdot \frac{1}{\cancel{n+1}} = \boxed{1 + \frac{1}{n}}$$

Bernoulli

2)  $b_n$  è decrescente

$$b_{n+1} < b_n \quad \text{?}$$

$$\left(1 + \frac{1}{n+1}\right)^{n+2} < \left(1 + \frac{1}{n}\right)^{n+1} \quad \text{?}$$

$$\parallel \left(\frac{n+2}{n+1}\right)^{n+2} \quad \parallel \left(\frac{n+1}{n}\right)^{n+1}$$

$$\left(\frac{n+1}{n+2}\right)^{n+2} > \left(\frac{n}{n+1}\right)^{n+1} \quad \text{?}$$

$$\left(\frac{n+2-1}{n+2}\right)^{\frac{n+2}{n+1}} > \left(\frac{n+1-1}{n+1}\right) = 1 - \frac{1}{n+1} \quad \text{?}$$

$$\left(1 - \frac{1}{n+2}\right)^{\frac{n+2}{n+1}} > 1 + \frac{n+2}{n+1} \cdot \left(-\frac{1}{n+2}\right) = 1 - \frac{1}{n+1}$$

Bernoulli

OSS. 1

$\forall n \in \mathbb{N} - \{0\}$

$$a_n < b_n \quad ?$$

$$\left(1 + \frac{1}{n}\right)^n \quad \left(1 + \frac{1}{n}\right)^{n+1}$$

OSS 2

$\forall i, j \in \mathbb{N} - \{0\}$

$$a_i < b_j$$

Caso particolare

$$a_{75} < b_{90}$$

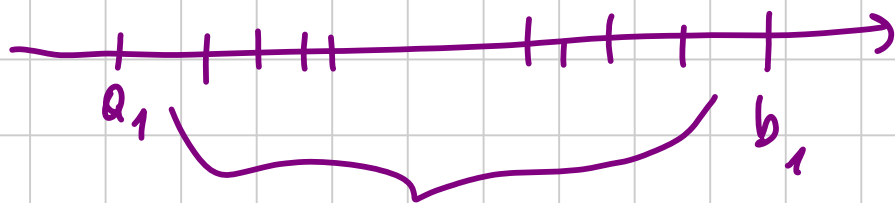
$$a_{75} < a_{90} < b_{90}$$

Prende  $k = \max\{i, j\}$

Avrà che

$$a_i \leq a_k < b_k \leq b_j$$

OSS 3 Tutti i termini delle 2 successioni sono nell'intervallo  $[a_1, b_1]$



OSS 4

Da ① e ② e OSS 3 segue che  $(a_n)$  e  $(b_n)$  sono monotone e limitate e quindi hanno limite finito.

$$a_n \rightarrow l$$

$$b_n \rightarrow L$$

③  $l = L$  (?)

$$b_n = \left(1 + \frac{1}{n}\right)^{n+1} = \underbrace{\left(1 + \frac{1}{n}\right)^n}_{a_n} \cdot \left(1 + \frac{1}{n}\right)^1$$

$$b_n = \left( 1 + \frac{1}{a_n} \right) \cdot a_n \Rightarrow L = e$$

$\downarrow$        $\downarrow$        $\downarrow$   
 $L$       1       $e$   
 $e$

Ultimo caso:

$$2 < l < 3$$

(51)

$$l > a_2 = \left( 1 + \frac{1}{2} \right)^2 = \left( \frac{3}{2} \right)^2 = \frac{9}{4} > 2$$

$$b_2 = \left( 1 + \frac{1}{2} \right)^3 = \left( \frac{3}{2} \right)^3 = \frac{27}{8}$$

$$b_5 = \left( 1 + \frac{1}{5} \right)^6 = \left( \frac{6}{5} \right)^6 = \dots < 3$$

**[Def.]** Indichiamo con "e" il valore del limite trovato nel T. 1

**[T. 2]** (Gener. di T. 1)

Se  $a_n \rightarrow +\infty$  allora  $\left( 1 + \frac{1}{a_n} \right)^{a_n} \rightarrow e$

Dim

$$\left(1 + \frac{1}{\lfloor a_n \rfloor + 1}\right)^{\lfloor a_n \rfloor} < \left(1 + \frac{1}{a_n}\right)^{a_n} < \left(1 + \frac{1}{\lfloor a_n \rfloor}\right)^{\lfloor a_n \rfloor + 1}$$
$$\left(1 + \frac{1}{\lfloor a_n \rfloor + 1}\right)^{\lfloor a_n \rfloor + 1} \cdot \left(1 + \frac{1}{\lfloor a_n \rfloor + 1}\right)^{-1}$$

Diagram showing the limit process for  $e$  using floor and ceiling functions. The first inequality shows  $\left(1 + \frac{1}{\lfloor a_n \rfloor + 1}\right)^{\lfloor a_n \rfloor} < \left(1 + \frac{1}{a_n}\right)^{a_n} < \left(1 + \frac{1}{\lfloor a_n \rfloor}\right)^{\lfloor a_n \rfloor + 1}$ . The second inequality shows  $\left(1 + \frac{1}{\lfloor a_n \rfloor + 1}\right)^{\lfloor a_n \rfloor + 1} \cdot \left(1 + \frac{1}{\lfloor a_n \rfloor + 1}\right)^{-1}$ . Arrows indicate the limit process leading to  $e$ .

ES. 1.7

$$\lim_{n \rightarrow +\infty} \left(1 + \frac{2}{n}\right)^n =$$

$$\left(1 + \frac{b_n}{n}\right)^n \xrightarrow{L} e^L$$

Diagram showing the limit process for  $e^L$  using the binomial expansion. The expression  $\left(1 + \frac{b_n}{n}\right)^n$  is shown with an arrow pointing to  $e^L$ .

$$= \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{\frac{n}{2}}\right)^{\frac{n}{2} \cdot 2} =$$

$$= \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{\frac{n}{2}}\right)^{\frac{n}{2}} \cdot \left(1 + \frac{1}{\frac{n}{2}}\right)^{\frac{n}{2}} = e^2$$

Diagram showing the limit process for  $e^2$  using the binomial expansion. The expression  $\left(1 + \frac{1}{\frac{n}{2}}\right)^{\frac{n}{2} \cdot 2}$  is shown with arrows pointing to  $e$  and  $e$ .

ES. 11.7

$$\lim_{h \rightarrow +\infty} \left(1 + \frac{1}{h^2}\right)^{2h^2+5} =$$

$$= \lim_{h \rightarrow +\infty} \underbrace{\left(1 + \frac{1}{h^2}\right)^{h^2}}_e \cdot \underbrace{\left(1 + \frac{1}{h^2}\right)^{h^2}}_e \cdot \underbrace{\left(1 + \frac{1}{h^2}\right)^5}_1 = e^2$$

E. 14.7

$$\lim_{h \rightarrow +\infty} \left(\frac{h+9}{h+5}\right)^{3h} =$$

$$= \lim_{h \rightarrow +\infty} \left(\frac{h+5+4}{h+5}\right)^{3h} =$$

$$= \lim_{h \rightarrow +\infty} \left(1 + \frac{1}{\frac{h+5}{4}}\right)^{\frac{h+5}{4} \cdot \frac{12h}{h+5}} = e^{12}$$

$$\frac{12h}{h+5} = \frac{12}{1+\frac{5}{h}} \rightarrow 12$$