Analisi Matematica 1 - Lezione 10 (I parte)

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Def. Date (a_n) e (b_n) tali che $a_n \to \mp\infty$ e $b_n \to \mp\infty$ diciamo che:

1) $a_n \approx b_n$ ("asintoticamente equivalente a") se $\frac{a_n}{b_n} \to 1$

2) a_n e b_n hanno lo stesso ordine di infinito se $\frac{a_n}{b_n} \to \ell$ finito e $\neq 0$

3) $a_n = o(b_n)$ se $\frac{a_n}{b_n} \to 0$

4) $a_n = O(b_n)$ se $\exists K > 0$ t.c. def. in n $\left|\frac{a_n}{b_n}\right| \leq K$

Regole

Teorema 1

Date (a_n), (b_n), (A_n) e (B_n) successioni che tendono a $+\infty$ t.c. $a_n \approx A_n$ e $b_n \approx B_n$ allora $\lim_{n\to+\infty} \frac{a_n}{b_n} = \lim_{n\to+\infty} \frac{A_n}{B_n}$

Dim

$$\lim_{n\to+\infty} \frac{a_n}{b_n} = \lim_{n\to+\infty} \underset{\substack{\downarrow \\ 1}}{\boxed{\frac{a_n}{A_n}}} \cdot \frac{A_n}{B_n} \cdot \underset{\substack{\downarrow \\ 1}}{\boxed{\frac{B_n}{b_n}}} =$$

Concludo che $\frac{a_n}{b_n} \to \ell \iff \frac{A_n}{B_n} \to \ell$

$\boxed{\text{Teorema 2}}$ Date (a_n), (b_n), (A_n) e (B_n) tutte che $\to \pm\infty$ e tali che

$$a_n = o(A_n) \quad e \quad b_n = o(B_n)$$

Allora:

$$\lim_{n\to+\infty} \frac{A_n + \overset{o(A_n)}{a_n}}{B_n + \underset{o(B_n)}{b_n}} = \lim_{n\to+\infty} \frac{A_n}{B_n}$$

$\boxed{\text{Dim}}$

$$\lim_{n\to+\infty} \frac{A_n + a_n}{B_n + b_n} = \lim_{n\to+\infty} \left(\frac{A_n}{B_n} \cdot \overset{\to 1}{\boxed{\frac{1 + \frac{a_n}{A_n}}{1 + \frac{b_n}{B_n}}}} \right)$$

quindi $\frac{A_n}{B_n} \to \ell \iff \frac{A_n + a_n}{B_n + b_n} \to \ell$

$\boxed{\text{Teorema 3}}$ (Solo $a > 1$) Date (a_n) e (b_n) entrambe $\to +\infty$ e tali che $\boxed{a_n \approx b_n}$. Allora

$$\log_a a_n \approx \log_a b_n$$

a_n e b_n hanno lo stesso ordine

Più debole

$\boxed{\text{Dim}}$ Devo mostrare che $\frac{\log_a a_n}{\log_a b_n} \to 1$.

$$\frac{\log_a a_n}{\log_a b_n} = \frac{\log_a\left(\frac{a_n}{b_n} b_n\right)}{\log_a b_n} = \frac{\log_a \frac{a_n}{b_n} + \log_a b_n}{\log_a b_n} =$$

$$= \underbrace{\frac{\boxed{\log_a\left(\frac{a_n}{b_n}\right)} \to 0}{\boxed{\log_a b_n} \to +\infty}}_{\to 0} + 1 = 1$$

$\boxed{\text{Esempio cattivo}}$

Se $a_n \approx b_n$ non è detto che $e^{a_n} \approx e^{b_n}$

infatti $a_n = n^2$ $\quad b_n = n^2 + n$

$\boxed{a_n \approx b_n}$ perché $\lim_{n \to +\infty} \frac{a_n}{b_n} = \lim_{n \to +\infty} \frac{n^2}{n^2+n} =$

$$= \lim_{n \to +\infty} \frac{n^2}{n^2 + o(n^2)} = \lim_{n \to +\infty} \frac{\cancel{n^2}}{\cancel{n^2}} = 1$$

Però $\lim_{n \to +\infty} \frac{e^{b_n}}{e^{a_n}} = \lim_{n \to +\infty} \frac{e^{n^2+n}}{e^{n^2}} =$

$$= \lim_{n \to +\infty} e^{(n^2+n)-n^2} = \lim_{n \to +\infty} e^n = +\infty$$

quindi: $e^{a_n} = o\left(e^{b_n}\right)$

$\boxed{\text{Teorema 4}}$ Siano (a_n) e (b_n) t.c. $\to +\infty$

Vogliamo capire cosa fa $\left(1+\frac{1}{a_n}\right)^{b_n}$

Allora si ha:

1) $a_n \simeq b_n \Rightarrow \left(1+\frac{1}{a_n}\right)^{b_n} \to e$

2) $\frac{b_n}{a_n} \to \alpha \Rightarrow \left(1+\frac{1}{a_n}\right)^{b_n} \to e^{\alpha}$

3) $a_n = o(b_n) \Rightarrow \left(1+\frac{1}{a_n}\right)^{b_n} \to +\infty$

4) $b_n = o(a_n) \Rightarrow \left(1+\frac{1}{a_n}\right)^{b_n} \to 1$

$\boxed{\text{Dim}}$ Dimostriamo ad esempio la (3).

$$\lim_{n\to+\infty}\left(1+\frac{1}{a_n}\right)^{b_n} = \lim_{n\to+\infty}\Bigg(\underbrace{\left(1+\frac{1}{a_n}\right)^{a_n}}_{e}\Bigg)^{\overbrace{\frac{b_n}{a_n}}^{+\infty}} = +\infty$$

$\boxed{\text{ESEMPI}}$

1) Conosciamo uno sbagliato

$$\left(1+\frac{1}{n}\right)^n \to e$$

OKKIO : SBAGLIATO

$$\left(1+\frac{1}{n}\right)^n = \left(1+o(1)\right)^n = 1^n \to 1$$

2) (Semplice)

$$\lim_{n\to+\infty}\left(\frac{e^n+n^3}{n^2+1} - \frac{e^n+n}{n^2+1}\right) = (*)$$

MODO GIUSTO

$$(*) = \lim_{n\to+\infty} \frac{\cancel{e^n}+n^3-\cancel{e^n}-n}{n^2+1} =$$

$$= \lim_{n\to+\infty} \frac{n^3 \underbrace{-n}_{o(n^3)}}{n^2 \underbrace{+1}_{o(n^2)}} =$$

$$= \lim_{n\to+\infty} \frac{n^3}{n^2} = \lim_{n\to+\infty} n = +\infty$$

$\boxed{\text{MODO SBAGLIATO}}$

$$\lim_{n\to+\infty}\left(\frac{e^n+o(e^n)}{n^2+1} - \frac{e^n+o(e^n)}{n^2+1}\right) =$$

ERRORE

$$= \lim_{n\to+\infty}\left(\frac{\cancel{e^n}}{n^2+1} - \frac{\cancel{e^n}}{n^2+1}\right) =$$

$$= 0$$

③

$$\lim_{n\to+\infty} \frac{\left(n^2+\frac{n^3+2}{n^2+1}\right)^{n^2}}{(n^2+1)^{n^2}} =$$

$$= \lim_{n\to+\infty} \frac{\left(n^2+o(n^2)\right)^{n^2}}{(n^2+1)^{n^2}} =$$

SBAGLIATO

$$= \lim_{n\to+\infty} \frac{(n^2)^{n^2}}{(n^2+1)^{n^2}} = \lim_{n\to+\infty} \frac{1}{\frac{(n^2+1)^{n^2}}{(n^2)^{n^2}}} =$$

$$= \lim_{n\to+\infty} \frac{1}{\left(1+\frac{1}{n^2}\right)^{n^2}} = \frac{1}{e} \quad \boxed{\text{NO}}$$

$\boxed{\text{MODO GIUSTO}}$

$$\lim_{n\to+\infty} \frac{\left(n^2+\frac{n^3+2}{n^2+1}\right)^{n^2}}{(n^2+1)^{n^2}} =$$

$$= \lim_{n\to+\infty} \frac{(n^2)^{n^2}}{(n^2+1)^{n^2}} \cdot \left(1+\frac{n^3+2}{n^4+n^2}\right)^{n^2} =$$

$$= \lim_{n\to+\infty} \frac{1}{\underbrace{\left(1+\frac{1}{n^2}\right)^{n^2}}_{\to e}} \cdot \underbrace{\left(1+\frac{1}{\frac{n^4+n^2}{n^3+2}}\right)^{n^2}}_{\to +\infty} = +\infty$$

$\frac{n^4+n^2}{n^3+2} \approx n \qquad a_n = o(n^2)$

$$\frac{\frac{n^4+n^2}{n^3+2}}{n} = \frac{n^4+n^2}{n^4+2n} \to 1$$

ESEMPIO 4

$$\boxed{e^n \approx 1+e^n}$$

$$\lim_{n\to+\infty} \frac{\ln(1+e^n)}{n+\sqrt{n}} = \lim_{n\to+\infty} \frac{\ln(e^n)}{n+\sqrt{n}}$$

$$= \lim_{n\to+\infty} \frac{n}{n+o(n)} = \lim_{n\to+\infty} \frac{\cancel{n}^{\,1}}{\cancel{n}_{\,1}} = 1$$

ESEMPIO 5

$$o(\sqrt{n+1}) = o(\sqrt{n})$$

ERRORE

$$\lim_{n\to+\infty} \frac{\sqrt{n+1}-\sqrt{n}+\ln n}{\ln(\ln n)} =$$

NO

$$= \lim_{n\to+\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\ln(\ln n)} = \lim_{n\to+\infty} \frac{1}{\underbrace{(\sqrt{n+1}+\sqrt{n})}_{+\infty}\cdot\underbrace{\ln(\ln n)}_{+\infty}} = 0$$

MODO GIUSTO

$$\lim_{n\to+\infty} \frac{\sqrt{n+1}-\sqrt{n}+\ln n}{\ln(\ln n)} =$$

$$= \lim_{n\to+\infty} \frac{\overbrace{(\sqrt{n+1}-\sqrt{n})}^{a_n} + \overbrace{\ln n}^{A_n}}{\ln(\ln(n))} =$$

$$= \lim_{n \to +\infty} \frac{(\ln n)}{\ln(\ln n)} = +\infty$$

$\boxed{\text{ESEMPIO 6}}$

$$\lim_{n \to +\infty} \left(1 + \frac{1}{(n+1)^n}\right)^{n^{n+1}} = +\infty$$

$(n+1)^n \to a_n$

$n^{n+1} \to b_n$

$$\frac{a_n}{b_n} = \frac{(n+1)^n}{n^{n+1}} = \underbrace{\left(1 + \frac{1}{n}\right)^n}_{\downarrow\, e} \cdot \underbrace{\frac{1}{n}}_{\downarrow\, 0} \to 0$$