

Analisi Matematica 1 - Lezione 18

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ESERCIZI SU LIMITI DI FUNZIONI

ES. 1

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{3^x - 2^x}{x} &= \lim_{x \rightarrow 0} \frac{(3^x - 1) + (1 - 2^x)}{x} = \\ &= \lim_{x \rightarrow 0} \left(\frac{e^{(\ln 3) \cdot x} - 1}{x} - \frac{e^{(\ln 2) \cdot x} - 1}{x} \right) = \\ &= \lim_{x \rightarrow 0} \left(\frac{x \cdot \ln 3}{x} - \frac{x \cdot \ln 2}{x} \right) = \boxed{\ln \frac{3}{2}} \end{aligned}$$

ES. 2

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{\frac{\pi}{2} - \operatorname{arctan} x}{1 - \cos \frac{1}{\sqrt{x}}} &= \\ &= \lim_{x \rightarrow +\infty} \frac{\boxed{\operatorname{arctan}\left(\frac{1}{x}\right)}}{\frac{1}{2} \cdot \left(\frac{1}{\sqrt{x}}\right)^2} \end{aligned}$$



$$\left[\begin{aligned} \frac{\pi}{2} - \operatorname{arctan} x &\neq \operatorname{arctan} \frac{1}{x} \\ \operatorname{arctan} x + \operatorname{arctan} \frac{1}{x} &\neq \frac{\pi}{2} \end{aligned} \right]$$

$$= \lim_{x \rightarrow +\infty} \frac{\boxed{\frac{1}{x}}}{\frac{1}{2} \left(\frac{1}{\sqrt{x}} \right)^2} =$$

$$= \lim_{x \rightarrow +\infty} \frac{\cancel{x}}{x} \cdot \frac{1}{\frac{1}{2} \cdot \cancel{x}} = 2$$

INTERMEZZO

$$\boxed{\lim_{x \rightarrow 0} \frac{\operatorname{arctan} x}{x} = 1} \quad \boxed{?}$$

$$\Rightarrow \lim_{y \rightarrow 0} \frac{y}{\tan y} = 1$$

$$\boxed{y = \operatorname{arctan} x} \Leftrightarrow \tan y = x$$

Se per $f(x) \rightarrow 0$ per $x \rightarrow x_0$
allora

$$\boxed{\lim_{x \rightarrow x_0} \frac{\operatorname{arctan}(f(x))}{f(x)} = 1}$$

↕

$$\boxed{\operatorname{arctan}(f(x)) \approx f(x) \text{ per } x \rightarrow x_0}$$

↕

$$\boxed{\operatorname{arctan}(f(x)) = f(x) + o(f(x)) \text{ per } x \rightarrow x_0}$$

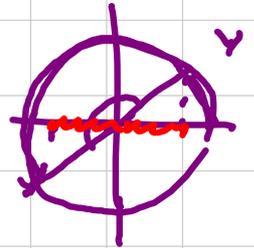
ES.3

$$\lim_{x \rightarrow \pi} \frac{1 + \cos x}{(\pi - x) \tan x} =$$

$$y = x - \pi$$

$$= \lim_{x \rightarrow \pi} \frac{1 + \cos(\pi + (x - \pi))}{(\pi - x) \tan(\pi + (x - \pi))}$$

$$= \lim_{y \rightarrow 0} \frac{1 + \cos(\pi + y)}{-y \cdot \tan(\pi + y)} =$$



$$= \lim_{y \rightarrow 0} \frac{1 - \cos y}{-y \cdot \tan y} =$$

$$= \lim_{y \rightarrow 0} \frac{\frac{y^2}{2}}{-y \cdot y} = -\frac{1}{2}$$

ES.4

$$\lim_{x \rightarrow 0^+} x^x = 1$$

Limite notevole

$$= \lim_{x \rightarrow 0^+} e^{\ln(x^x)} =$$

$$= \lim_{x \rightarrow 0^+} e^{x \ln(x)} = e^0 = 1$$

$$\lim_{x \rightarrow 0^+} -x \ln(x) = \lim_{x \rightarrow 0^+} \frac{\ln\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} \stackrel{y = \frac{1}{x}}{\downarrow} = \lim_{y \rightarrow +\infty} \frac{\ln y}{y} = 0$$

ES 5

$$\lim_{x \rightarrow 0^+} \frac{x^x - 1}{x} =$$

$$= \lim_{x \rightarrow 0^+} \frac{e^{\boxed{x \ln x}} - 1}{x} =$$

$$= \lim_{x \rightarrow 0^+} \frac{x \ln x}{x} =$$

$$= -\infty$$

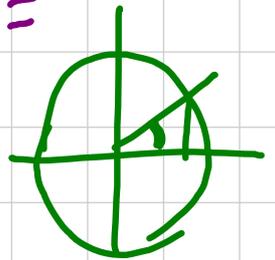
(perché so che
 $x \ln x \rightarrow 0$
 per $x \rightarrow 0^+$)

ES 6

$$\lim_{x \rightarrow 0^+} \frac{(\sin x)^{\sin x} - 1}{x} =$$

$$= \lim_{x \rightarrow 0^+} \frac{e^{\boxed{\sin x \cdot \ln(\sin x)}} - 1}{x} =$$

$$= \lim_{x \rightarrow 0^+} \frac{\sin x \cdot \ln(\sin x)}{x} =$$



$$= \lim_{x \rightarrow 0^+} \left(\frac{\sin x}{x} \right) \cdot \ln(\sin x) = -\infty$$

\downarrow \downarrow
 1 $-\infty$

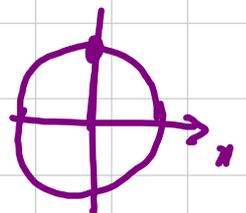
$$\lim_{x \rightarrow 0^+} \sin x \cdot \ln(\sin x) =$$

$$= \lim_{y \rightarrow 0^+} y \ln y = 0$$

\uparrow \uparrow
 $y = \sin x$ FATTO PRIMA

ES 7

$$\lim_{x \rightarrow \frac{\pi}{2}^+} (\sin x)^{\frac{1}{\cos x}} =$$



$$\begin{aligned}
 &= \lim_{x \rightarrow \frac{\pi}{2}^+} \left(\sin\left(\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right)\right) \right)^{\frac{1}{\cos\left(\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right)\right)}} = \\
 &\stackrel{\boxed{y = x - \frac{\pi}{2}}}{=} \lim_{y \rightarrow 0^+} \left(\sin\left(\frac{\pi}{2} + y\right) \right)^{\frac{1}{\cos\left(\frac{\pi}{2} + y\right)}} = \\
 &= \lim_{y \rightarrow 0^+} \left(\cos(-y) \right)^{\frac{1}{\sin(-y)}} = \\
 &= \lim_{y \rightarrow 0^+} \left(\cos y \right)^{-\frac{1}{\sin y}} = \\
 &= \lim_{y \rightarrow 0^+} \left(1 + \boxed{\cos y - 1} \right)^{-\frac{1}{\sin y}} = \boxed{\lim_{x \rightarrow 0} \left(1 + \boxed{x} \right)^{\frac{1}{\boxed{x}}}} = e \\
 &= \lim_{y \rightarrow 0^+} \left(1 + (\cos y - 1) \right)^{\frac{1}{\cos y - 1}} \cdot \frac{-\cos y + 1}{\sin y} \rightarrow 0 = e^0 = 1 \\
 &\quad \downarrow \\
 &\quad \boxed{e}
 \end{aligned}$$

$$\boxed{\text{ES. 8}} \quad \lim_{x \rightarrow -\infty} \frac{\sigma(x) \cdot e^x - x}{\sqrt{1+x^2} - x} =$$

$$= \lim_{x \rightarrow -\infty} \frac{-x}{\boxed{|x|} \sqrt{\frac{1}{x^2} + 1} - x} =$$

$$= \lim_{x \rightarrow -\infty} \frac{\cancel{-x}}{\cancel{-x} \left(\sqrt{1 + \frac{1}{x^2}} + 1 \right)} = \frac{1}{2}$$

ES. 9 $\lim_{x \rightarrow 0} \frac{(1+3x)^{8x} - 1}{(1+4x)^{6x} - 1} =$

$$= \lim_{x \rightarrow 0} \frac{e^{\boxed{8x \cdot \ln(1+3x)}} - 1}{e^{6x \cdot \ln(1+4x)} - 1} =$$

$$= \lim_{x \rightarrow 0} \frac{\cancel{8x} \cdot \ln(1+3x)}{\cancel{6x} \cdot \ln(1+4x)} =$$

$$= \lim_{x \rightarrow 0} \frac{\cancel{8} \cdot 3x}{\cancel{6} \cdot 4x} = 1$$

ES. 10

$$\lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x^x} \right)^{(x+1)^x} =$$

$$= \lim_{x \rightarrow +\infty} \left(\left(1 + \frac{1}{x^x} \right)^{x^x} \right)^{\frac{(x+1)^x}{x^x}} = e^{\rightarrow}$$

$$= \lim_{x \rightarrow +\infty} \left(\boxed{\left(1 + \frac{1}{x^x} \right)^{x^x}} \right)^{\boxed{\left(1 + \frac{1}{x} \right)^x}} = e^e$$

ES. 11

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^2} = 0$$

??

$$\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3} = \frac{1}{2}$$

Dim.

$$| \tan x - x | < | \tan x - \sin x |$$

$x \text{ con } |x| < \frac{\pi}{2}$

$$0 < x < \frac{\pi}{2}$$

$0 < \sin x < x < \tan x$

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^2} =$$

per il teorema di de L'Hôpital $\alpha = 3 - 2 = 1$

$$= \lim_{x \rightarrow 0} \frac{\tan x - x}{\tan x - \sin x} \cdot \frac{\tan x - \sin x}{x^2} = 0$$

limitato

Se $\alpha = 3$

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3}$$

non posso dire che vale 0

non vale 0

$$\left| \frac{\tan x - x}{x^3} \right| < K$$

per qualche costante $K > 0$

Cisä

$$\tan x - x = \mathcal{O}(x^3) \quad \text{für } x \rightarrow 0$$

$$\boxed{\tan x = x + \mathcal{O}(x^3)}$$

$$\tan x = x + \sigma(x)$$