Analisi Matematica 1 - Lezione 28

Titolo nota 1 Dicembre 2014 - docente: Prof. **Emanuele Callegari** - Università di Roma ***Tor Vergata***

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SVILUPPI DI TAYLOR DI F. PARTICOL.

$\boxed{\text{ES 1}}$ e^x $\quad x_0 = 0$

$$T_n[e^x](x) = f(0) + f'(0)\cdot x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

$$\boxed{D^n(e^x) = e^x}$$

$$\boxed{T_n[e^x](x)} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

$$e^x - \circ = o(x^n)$$

$$\boxed{\boxed{e^x = 1 + x} \underbrace{+ \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^k}{k!} + \dots + \frac{x^n}{n!} + o(x^n)}_{o(x)}}$$

$\boxed{\text{ES 2}}$ $f(x) = \sin x \quad x_0 = 0 \quad$ fino a ordine $2n+1$

$$T_{2n+1}[\sin x](x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots + \frac{f^{(2n+1)}(0)}{(2n+1)!}x^{2n+1}$$

$$f(x) = \sin x = f^{(4)}(x) \qquad f(0) = 0$$

$$f'(x) = \cos x = f^{(5)}(x) \qquad f'(0) = 1 \leftarrow$$

$$f''(x) = -\sin x \qquad f''(0) = 0$$

$$f'''(x) = -\cos x \qquad f'''(0) = -1$$

$$\boxed{\sin x = x} \underbrace{- \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + O(x^{2n+3})}_{o(x)}$$

ES 3

$$\boxed{\cos x = 1 - \frac{x^2}{2!}} \underbrace{+ \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + O(x^{2n+2})}_{o(x^2)}$$

(Per Casa)

ES 4

$$\frac{1}{1-x} = f(x) \qquad x = 0 \qquad n$$

$$\frac{1}{1-x} - P(x) = o(x^n)$$

?

IDEA

$$(1-x)(1+x) = 1 - x^2$$

$$(1-x)(1+x+x^2) = 1 - x^3$$

$$\vdots$$

$$(1-x)(1+x+x^2+\dots+x^n) = 1 + \cancel{x} + \cancel{x^2} + \cancel{\dots} + \cancel{x^n} - \cancel{x} - \cancel{x^2} - \cancel{x^3} - \dots - x^{n+1} =$$

$$= 1 - x^{n+1}$$

$$\underline{1 + x + x^2 + \dots + x^n} = \frac{1 - x^{n+1}}{1-x} = \boxed{\frac{1}{1-x}} - \boxed{\frac{x^{n+1}}{1-x}}$$

$o(x^n)$

$$\frac{1}{1-x} - \left(1+x+x^2+\cdots+x^n\right) = \frac{x^{n+1}}{1-x} = o(x^n) = O(x^{n+1})$$

$$\frac{1}{1-x} = 1+x+x^2+\cdots+x^n+o(x^n)$$

$\boxed{\text{ES 5}}$ $f(x) = \dfrac{1}{1+x}$ $\quad x_0 = 0 \quad$ ordine n

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1+(-x)+(-x)^2+(-x)^3+\cdots+(-x)^n+o(x^n)$$

se $g(x) \to 0$

$$\frac{1}{1-g(x)} = \underbrace{1+g(x)+(g(x))^2+(g(x))^3+\cdots+(g(x))^n}_{P(g(x))}+o((g(x))^n)$$

$$\frac{1}{1-g(x)} - P(g(x)) = o\left((g(x))^n\right) \quad ?$$

$$\Updownarrow$$

$$\frac{\frac{1}{1-g(x)} - P(g(x))}{(g(x))^n} \longrightarrow 0$$

sì per cambio variabile

$\boxed{y = g(x)}$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + o(x^n)$$

$\boxed{ES\,6}$ $f(x) = \frac{1}{1+x^2}$ $x_0 = 0$ $2n$

$$\boxed{\frac{1}{1+x^2}} = \frac{1}{1+(x^2)} = 1 - x^2 + x^4 - x^6 + \dots + (-1)^n x^{2n} + o(x^{2n+2})$$

$\boxed{E\,7}$ $\ln(1+x)$ $x=0$ n

$\boxed{OSS:}$ $\left(T_n[f]\right)' = T_{n-1}[f']$

$\boxed{Dim}$

$$T_n[f] = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$T_{n-1}[f'] = f'(x_0) + f''(x_0)(x-x_0) + \frac{f'''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{(n-1)!}\cdot(x-x_0)^{n-1}$$

$\boxed{SI}$

$f(x) = \ln(1+x)$ $f'(x) = \frac{1}{1+x}$

$$T_{n-1}\left[\frac{1}{1+x}\right] = 1 - x + x^2 - x^3 + \dots + (-1)^{n-1} x^{n-1}$$

$$T_n\left[\ln(1+x)\right] = \boxed{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n-1}}{n}x^n}$$

$$\boxed{\boxed{\ln(1+x) = x} \underbrace{- \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots + \frac{(-1)^{n-1}}{n}x^n + o(x^n)}_{o(x)}}$$

$\boxed{\text{ES 8}}$ $\quad f(x) = \arctan x \qquad x = 0$

$$f'(x) = \frac{1}{1+x^2}$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + o(x^{2n+2})$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + o(x^{2n+3})$$

$\boxed{\text{ES. 9}}$ $\quad (1+x)^\alpha = f(x) \qquad \boxed{x_0 = 0}$

$$(1+x)^\alpha = 1 + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \cdots + \binom{\alpha}{n}x^n + o(x^n)$$

$$\boxed{\binom{\alpha}{k}} = \frac{\overbrace{\alpha\cdot(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}^{k}}{k!}$$

Caso particolare $\alpha = \frac{1}{2}$

$$(1+x)^{\frac{1}{2}} = \sqrt{1+x} = 1 + \binom{\frac{1}{2}}{1} x + \binom{\frac{1}{2}}{2} x^2 + o(x^2)$$

$$= 1 + \frac{\frac{1}{2}}{1!} x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!} x^2 + o(x^2)$$

$$= 1 + \frac{1}{2} x - \frac{1}{8} x^2 + o(x^2)$$

$o(x)$

$$\boxed{\sqrt{1+x} = 1 + \frac{1}{2} x} - \frac{1}{8} x^2 + o(x^2)$$

Esercizi

ES. 1 Trovare Pol. di Taylor di ordine 13 di $\boxed{\sin(\boxed{3x^2})}$ in $x_0 = 0$

$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} + O(y^7)$$

$$\sin(3x^2) = 3x^2 - \frac{(3x^2)^3}{3!} + \frac{(3x^2)^5}{5!} + O\left((3x^2)^7\right) =$$

$$= \boxed{3x^2 - \frac{9}{2} x^6 + \frac{3^5}{5!} x^{10}} + o(x^{13})$$

$\boxed{ES.2}$ $\quad (x-\sin x)\left(\ln(1-x)+\ln(1+x)\right) = f(x)$

Trovare pol. di T. di ordine 7 in $x_0=0$

$$x-\sin x = \cancel{x} - \left(\cancel{x} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + o(x^9)\right) =$$

$$= \boxed{\frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} + o(x^9)} = \boxed{\frac{x^3}{3!} - \frac{x^5}{5!} + o(x^6)}$$

$\ln(1+(-x))$

$$\ln(1-x)+\ln(1+x) = \cancel{(-x)} - \frac{(-x)^2}{2} + \cancel{\frac{(-x)^3}{3}} - \frac{(-x)^4}{4} + \cancel{\frac{(-x)^5}{5}} - \frac{(-x)^6}{6} + \cancel{\frac{(-x)^7}{7}}$$

$$+ o(x^7) + \cancel{x} - \frac{x^2}{2} + \cancel{\frac{x^3}{3}} - \frac{x^4}{4} + \cancel{\frac{x^5}{5}} - \frac{x^6}{6} + \cancel{\frac{x^7}{7}} + o(x^7) =$$

$$= \boxed{\boxed{-x^2 - \frac{x^4}{2}} - \frac{x^6}{3} + o(x^7)} =$$

$$= \boxed{-x^2 - \frac{x^4}{2} + o(x^6)}$$

$$\left(\frac{x^3}{3!} - \frac{x^5}{5!} + o(x^6)\right)\left(-x^2 - \frac{x^4}{2} + o(x^6)\right) =$$

$$= -\frac{x^5}{3!} - \frac{x^7}{2\cdot 3!} + o(x^9) + \frac{x^7}{5!} =$$

$$= \boxed{-\frac{x^9}{6} + \left(\frac{1}{5!} - \frac{1}{2\cdot 3!}\right) x^7} + O(x^9)$$