Università di Roma Tor Vergata - Facoltà di ingegneria

Corso di Analisi Matematica II per ingegneria Gestionale

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Lezione 3: SERIE NUMERICHE (III parte)

Criterio della Radice

Data $\sum_{n=0}^{+\infty} a_n$ a termini positivi allora:

→ 1) se $\sqrt[n]{a_n} \geq 1$ def. in n allora $\sum a_n$ diverge

2) se $\underbrace{\sqrt[n]{a_n} \leq \lambda < 1}$ def. in n allora $\sum a_n$ converge

$\boxed{\text{Dim}}$

① $\Rightarrow a_n \geq 1 \Rightarrow a_n \not\to 0 \Rightarrow \sum a_n$ diverge a $+\infty$

② $\Rightarrow$ def. in n: $a_n \leq \lambda^n$ (con $\lambda < 1$)

$\sum \lambda^n$ converge $\dfrac{1}{1-\lambda}$

$\sum a_n$ converge

per confronto con

$\boxed{\text{Esempio 1}}$ $\displaystyle\sum_{n=1}^{+\infty} \left(1-\frac{1}{n}\right)^{n^2}$

a_n

$$\underbrace{\left(1-\frac{1}{n}\right)^{n^2}} < \underbrace{\left(\frac{1}{2}\right)^{n}}$$

$$\sqrt[n]{a_n} = \sqrt[n]{\left(1-\frac{1}{n}\right)^{n^2}} = \left(1-\frac{1}{n}\right)^{n} \longrightarrow \frac{1}{e} < 1$$

$\boxed{\text{Criterio del Rapporto}}$ Data $\sum_{n=0}^{+\infty} a_n$ a termini positivi, allora

$\Rightarrow$ 1) se $\frac{a_{n+1}}{a_n} \geq 1$ def. in n allora $\sum a_n$ diverge

$\rightarrow$ 2) se invece $\exists \lambda$ t.c. $\boxed{\frac{a_{n+1}}{a_n} \leq \lambda < 1}$ def. in n allora $\sum a_n$ converge.

$\boxed{\text{Dim}}$ 1 $\Rightarrow$ def. in n $a_{n+1} \geq a_n$

da n_0 in poi cresce quindi ogni a_n con $n \geq n_0$ è $\geq a_{n_0}$ $\Rightarrow$ $a_n \not\to 0$ $\Rightarrow \sum a_n$ diverge

② Senza perdere di generalità posso supporre che $\frac{a_{n+1}}{a_n} \le \lambda < 1$ sempre.

$$a_1 \le \lambda a_0$$

$$a_2 \le \lambda a_1 \le \lambda^2 a_0$$

$$a_3 \le \lambda a_2 \le \lambda^3 a_0$$

$$\vdots$$

$$\boxed{a_n \le \lambda^n a_0}$$

$$\frac{a_{n+1}}{a_n} \le \lambda \iff a_{n+1} \le \lambda a_n$$

$\sum a_n$ converge

$\sum \lambda^n \cdot a_0$

$a_0 \cdot \sum \lambda^n$ converge

$\boxed{\text{Esempio 2}}$

$$\sum_{n=0}^{+\infty} \frac{1}{n!}$$

a_n

$$\boxed{\frac{a_{n+1}}{a_n}} = \frac{\frac{1}{(n+1)\cancel{!}}}{\frac{1}{\cancel{n!}}} = \frac{1}{n+1} \to 0$$

$\boxed{\text{Esempio 3}}$

$$\sum_{n=0}^{+\infty} \frac{n!}{n^n} \qquad a_n = \frac{n!}{n^n}$$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{(n+1)}\cancel{!}}{(n+1)^{n+\cancel{1}}} \cdot \frac{n^n}{\cancel{n!}} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n =$$

$$= \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \longrightarrow \frac{1}{e}.$$

$\boxed{\text{Exemplu 4}}$ $\displaystyle\sum_{n=1}^{+\infty} \underbrace{\frac{n!}{n^n} a^n}_{a_n}$ $\boxed{a > 0}$

$$\boxed{\frac{a_{n+1}}{a_n}} = \frac{\cancel{(n+1)!}\, a^{\cancel{n+1}}}{\underbrace{(n+1)^{n+\cancel{1}}}} \cdot \frac{n^n}{\cancel{n!} \cdot \cancel{a^n}} = a \cdot \frac{n^n}{(n+1)^n} =$$

$$= \frac{a}{\left(\frac{n+1}{n}\right)^n} = \frac{a}{\boxed{\left(1+\frac{1}{n}\right)^n}} = \boxed{\frac{a}{e}}$$

$\boxed{a>e}$ $\boxed{\dfrac{a_{n+1}}{a_n}}\geq 1$ def. in n $\Longrightarrow$ $\boxed{\sum a_n \text{ diverge}}$

$\boxed{a<e}$ $\dfrac{a_{n+1}}{a_n} \to \dfrac{a}{e} < 1 \Longrightarrow$ $\boxed{\sum a_n \text{ converge}}$

$\boxed{a=e}$ $\dfrac{a_{n+1}}{a_n} = \boxed{\dfrac{e}{\left(1+\frac{1}{n}\right)^n}} > 1$ $\boxed{\sum a_n \text{ diverge}}$

$(a>0)$ Converge se e solo se $\boxed{a<e}$

$\boxed{\text{Exemple 5}}$ $\displaystyle\sum_{n=1}^{+\infty}\left(e-\left(1+\frac{1}{n}\right)^{n}\right)$

$\boxed{e^{x}-1}\approx x$

$$\boxed{e-\left(1+\frac{1}{n}\right)^{n}} = e-e^{\ln\left(1+\frac{1}{n}\right)^{n}} = e-e^{n\ln\left(1+\frac{1}{n}\right)} =$$

$$\boxed{\frac{e}{2n}}$$

$$\boxed{x=e^{\ln(x)}} \quad x>0$$

$$= -e\left(e^{\boxed{n\ln\left(1+\frac{1}{n}\right)-1}}-1\right) \approx -e\cdot\boxed{} \approx -e\left(-\frac{1}{2n}\right)$$

$$\approx \boxed{\frac{e}{2n}}$$

$$\ln(1+x) = x - \frac{x^2}{2} + o(x^2)$$

$$\boxed{n\boxed{\ln\left(1+\frac{1}{n}\right)}-1} = n\left(\frac{1}{n} - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)\right) - 1 =$$

$$= \cancel{1} - \frac{1}{2n} + o\left(\frac{1}{n}\right) \cancel{-1} = \boxed{-\frac{1}{2n} + o\left(\frac{1}{n}\right)}$$

$$\approx \boxed{-\frac{1}{2n}} +$$

$$\sum_{n=1}^{+\infty} \frac{e}{n} = e \cdot \sum_{n=1}^{+\infty} \frac{1}{n} \quad \boxed{\text{diverge}}$$

$\boxed{\text{Esempio 6}}$

$$\sum_{n=1}^{+\infty} \boxed{\left(\boxed{e - \left(1+\frac{1}{n}\right)^n}\right)^a} \quad \left(\begin{array}{l}\text{termini positivi} \\ a>0 \text{ converge}\end{array}\right)$$

$$\wr\wr$$

$$\boxed{\frac{e}{2n}} \qquad \boxed{a > 1}$$

$$\sum_{n=1}^{+\infty} \boxed{\frac{e^a}{2^a} \cdot \frac{1}{n^a}} = \frac{e^a}{2^a} \boxed{\sum_{n=1}^{+\infty} \frac{1}{n^a}}$$