Università di Roma Tor Vergata - Facoltà di ingegneria

Corso di Analisi Matematica II per ingegneria Gestionale

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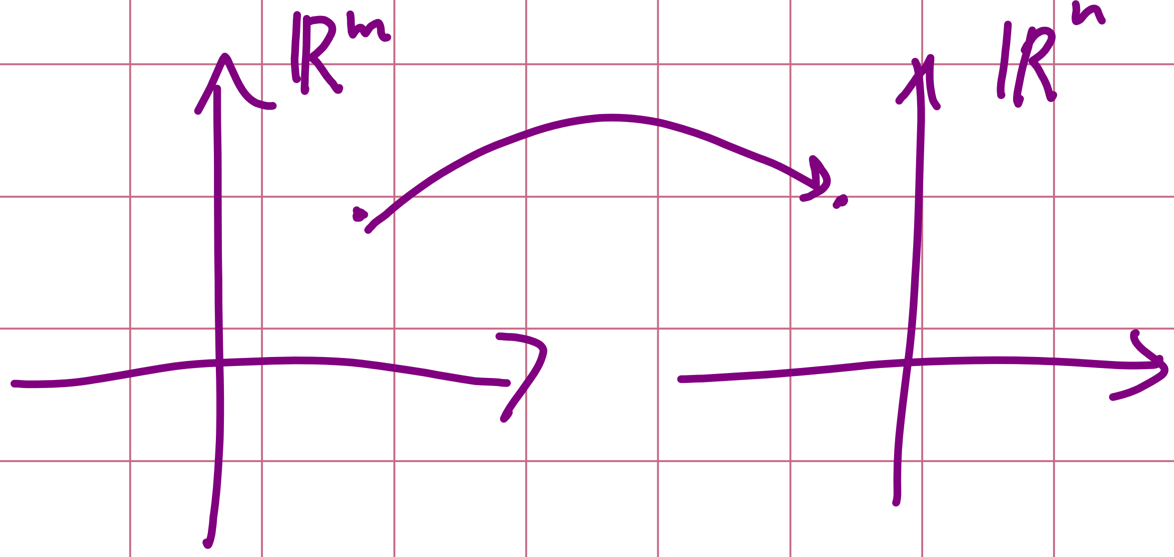
Lezione 6: LIMITI IN $\mathbb{R}^n$ (I Parte)

$f : \mathbb{R}^m \to \mathbb{R}^n$

$f : \Omega \to \mathbb{R}^n$

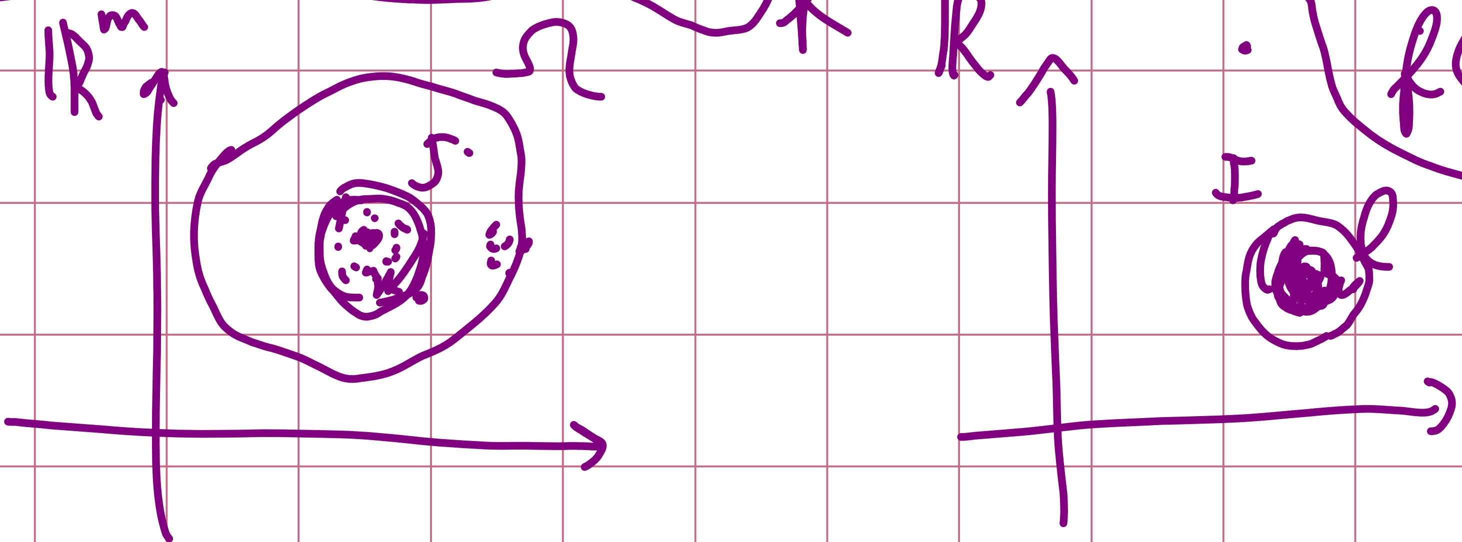
$\Omega \subset \mathbb{R}^m$



$\boxed{\text{Def}}$ Sia $\Omega \subset \mathbb{R}^m$, x_0 pt. di acc. per Ω e $f: \Omega \to \mathbb{R}^n$

Diremo che "$\lim_{x \to x_0} f(x) = \ell \in \mathbb{R}^n$" se

"$\forall\, I$ intorno di ℓ $\exists\, J$ intorno di x_0 t.c. $\forall x \in (J \cap \Omega) - \{x_0\}$ $f(x) \in I$"

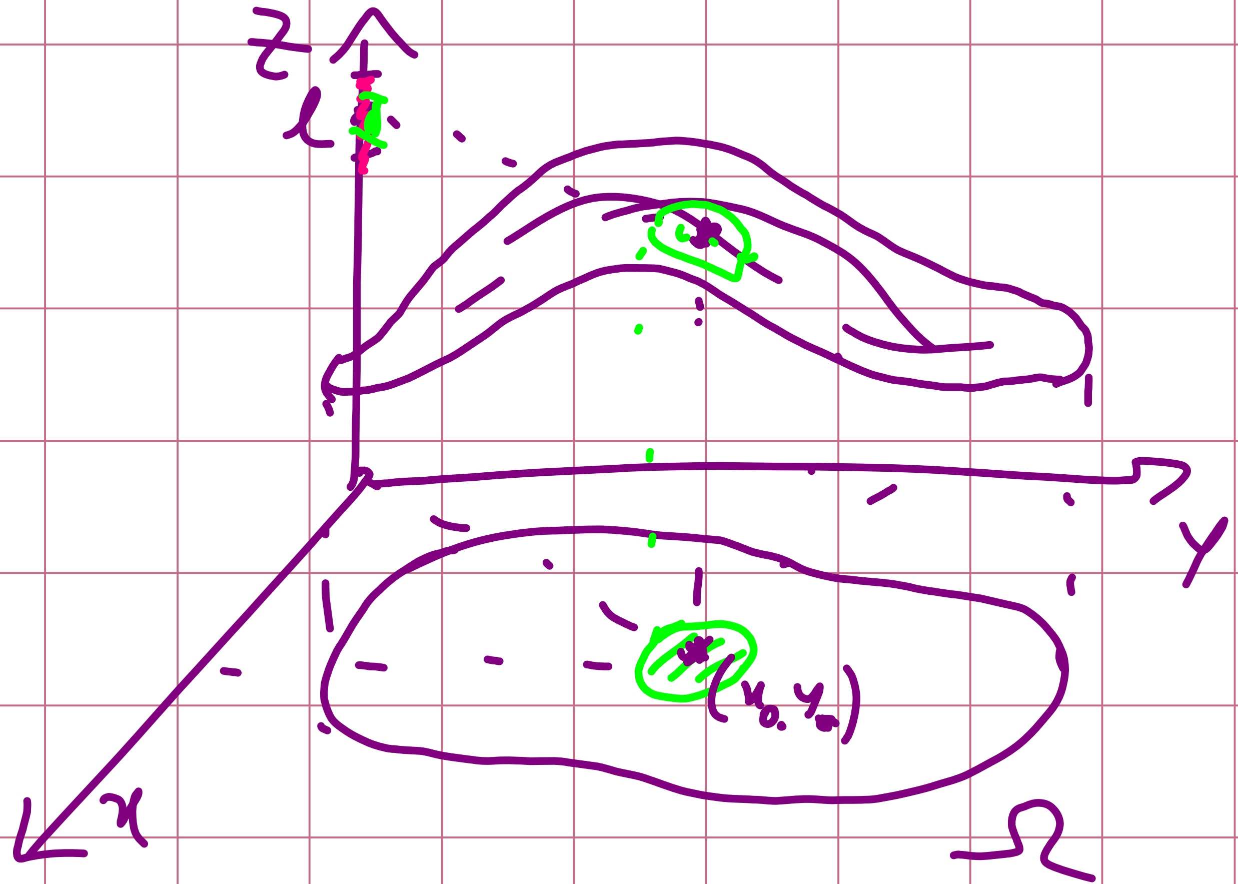


$\boxed{\text{Def}}$ Dati $\Omega \subset \mathbb{R}^m$, $x_0 \in \Omega$ e $f : \Omega \to \mathbb{R}^n$, diremo che f è <u>continua</u> in x_0 se

"$\forall I$ intorno di $f(x_0)$ $\exists J$ intorno di x_0 t.c. $\forall x \in J \cap \Omega$

si ha $f(x) \in I$"

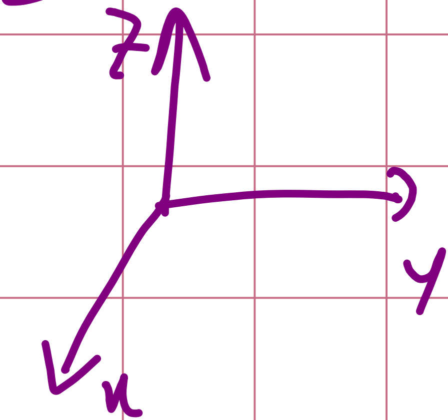
Caso particolare $m=2$ e $n=1$

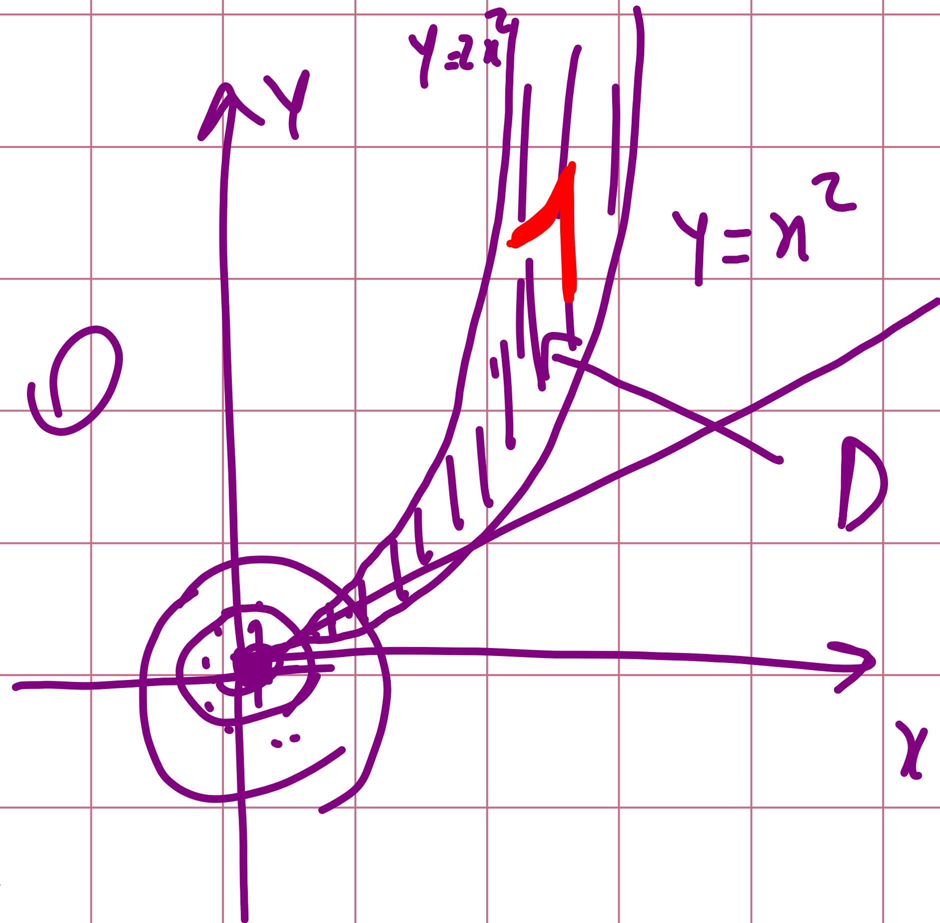


$$(x, y, f(x,y))$$

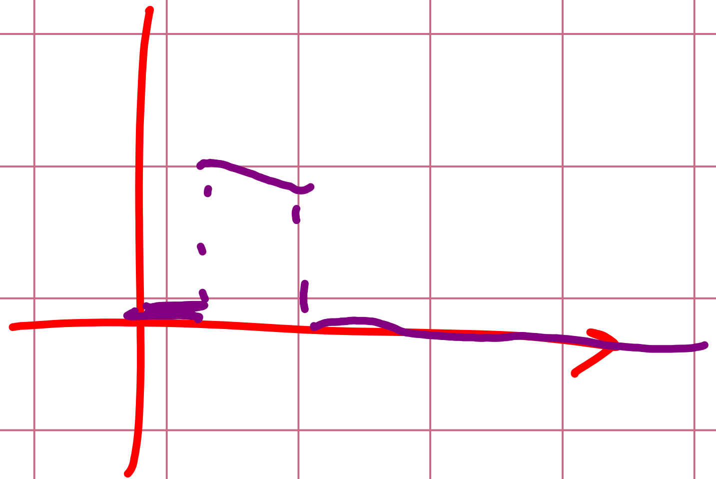
$$(x,y) \in \Omega$$

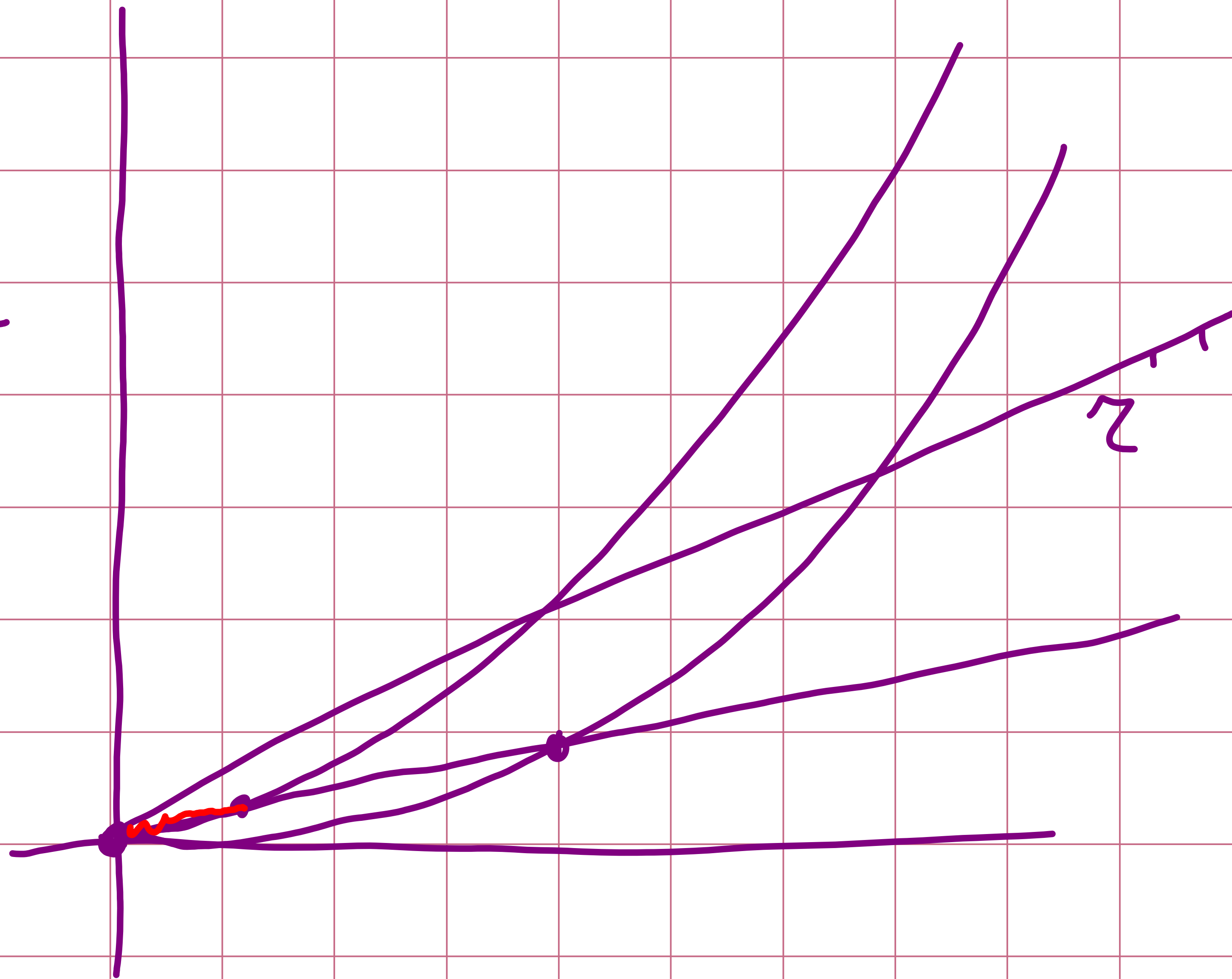
$\boxed{\text{Exemplul 1}}$





$$f(x,y) = \begin{cases} 1 & \text{u } (x,y) \in D \\ 0 & \text{altminteri} \end{cases}$$

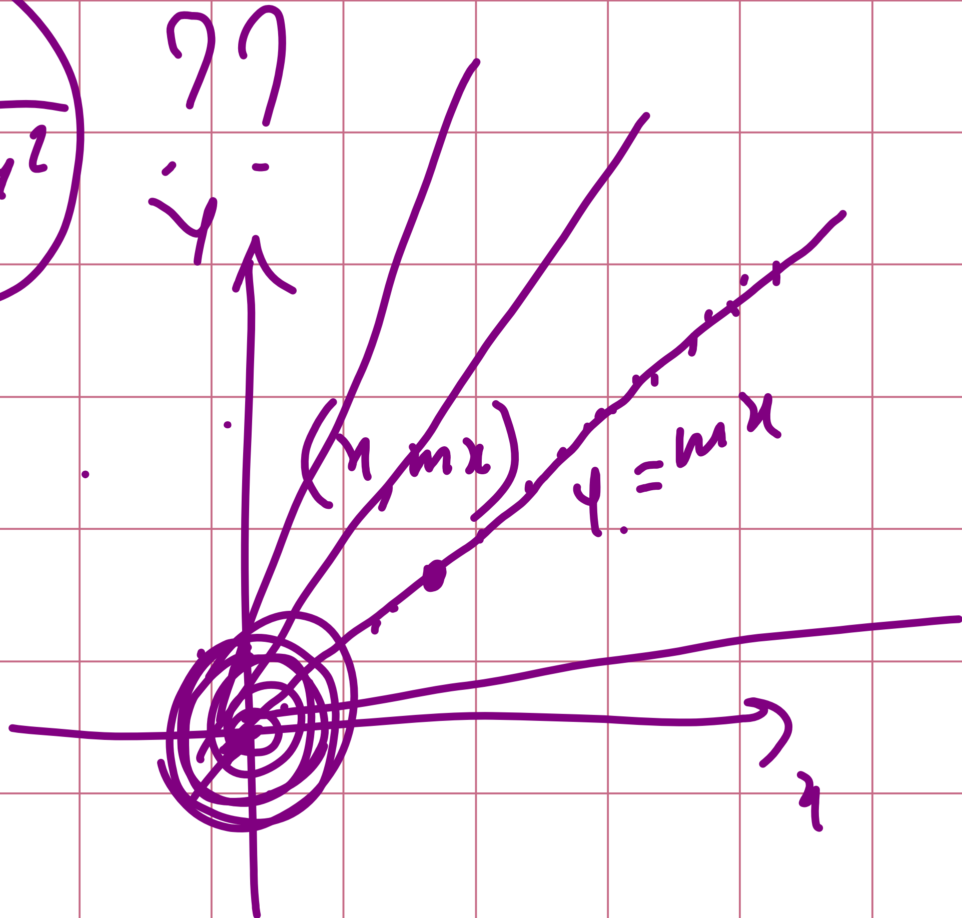




Exemple 2

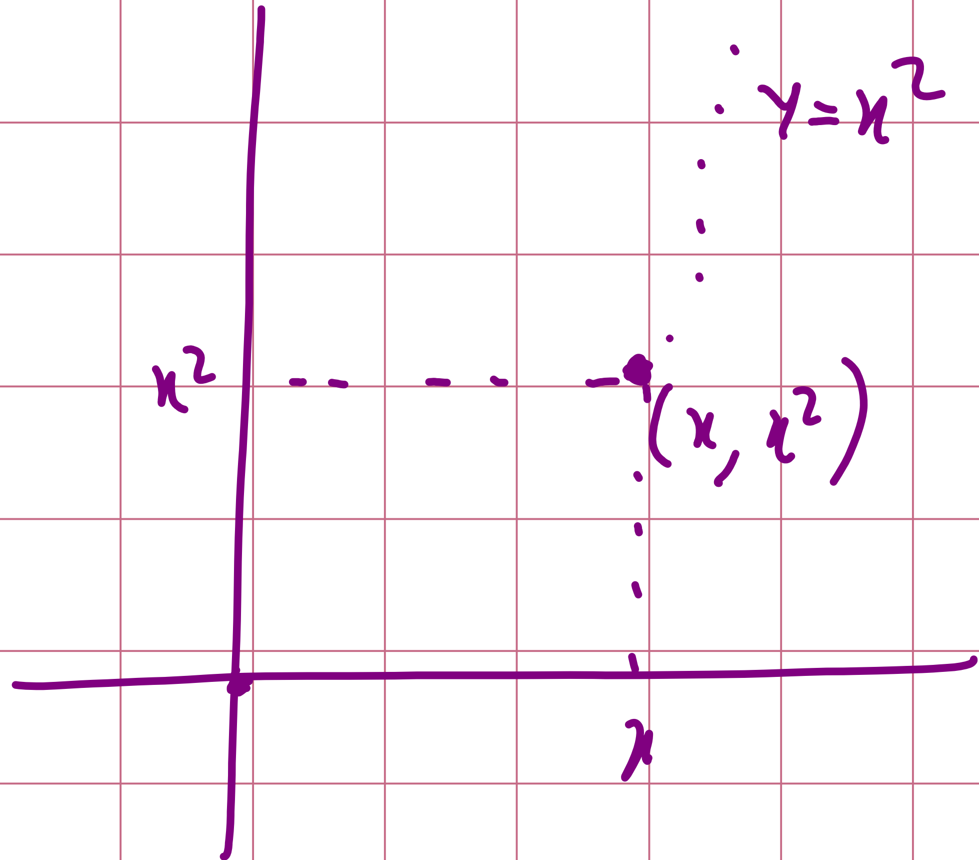
$$f(x,y) = \frac{x^2 y}{x^4 + y^2} \qquad (x,y) \neq (0,0)$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2} \quad ??$$



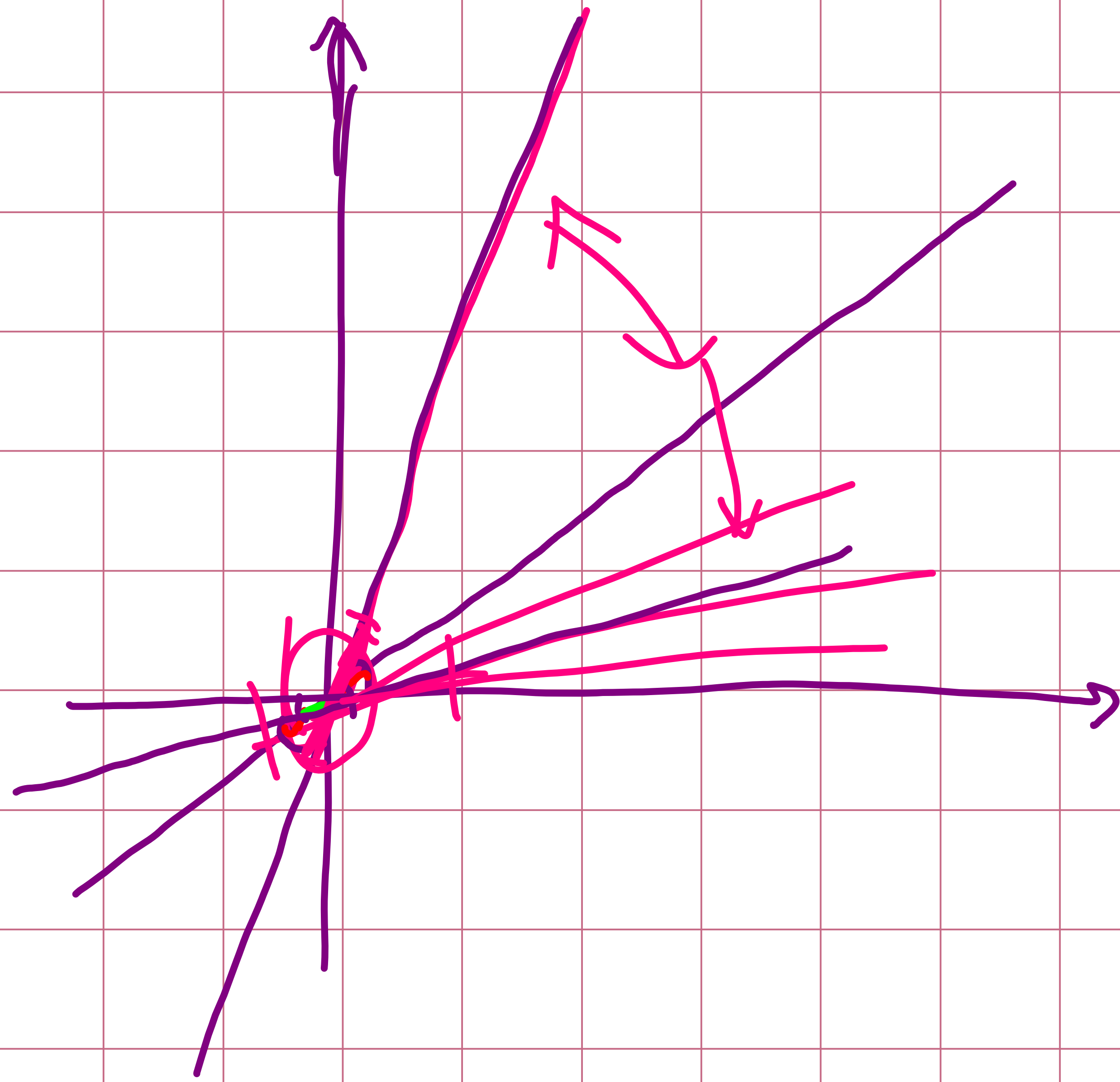
$$\lim_{x \to 0} \frac{x^2 \cdot mx}{\underbrace{x^4}_{} + \underbrace{m^2 x^2}_{}} = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$$



$$\lim_{x\to 0} \frac{x^2\cdot(x^2)}{x^4+(x^2)^2} =$$

$$= \lim_{x\to 0} \frac{\cancel{x^4}}{2\cancel{x^4}} = \frac{1}{2}$$



$\boxed{\text{Teorema delle operazioni sui limiti}}$

Dato $\Omega \in \mathbb{R}^n$, x_0 pt. di acc. per Ω, $f, g : \Omega \to \mathbb{R}$ tali che $\lim_{x \to x_0} f(x) = l_1$ e $\lim_{x \to x_0} g(x) = l_2$. Allora

1) $\lim_{x \to x_0} f(x) + g(x) = l_1 + l_2$

2) $\lim_{x \to x_0} \alpha f(x) = \alpha l_1$

3) $\lim_{x \to x_0} f(x) \cdot g(x) = l_1 \cdot l_2$ (se $l_2 = 0$ basta che f sia limitata

4) $\lim_{x \to x_0} \boxed{\dfrac{f(x)}{g(x)}} = \dfrac{l_1}{l_2}$ se $l_2 \neq 0$

Teorema (del confronto)

Dati $\Omega \subset \mathbb{R}^n$, x_0 di acc. per Ω e $f, g, h : \Omega \to \mathbb{R}$

t.c. $\lim_{x \to x_0} f(x) = \ell = \lim_{x \to x_0} h(x)$.

Supponiamo inoltre che $f(x) \leq g(x) \leq h(x) \quad \forall x \in \Omega$.

Allora anche $\lim_{x \to x_0} g(x) = \ell$.

<u>Teorema</u> $\boxed{\text{limite delle restrizioni}}$

Dati $\Omega \subset \mathbb{R}^n$, x_0 pt di acc. per Ω e $f : \Omega \to \mathbb{R}$

t.c. $\lim_{x \to x_0} f(x) = \ell$. Allora, preso comunque

$K \subset \Omega$ avente ancora x_0 come punto di accumulazione, si ha che

$$\lim_{x \to x_0} f\big|_K(x) = \ell$$

Exemple 1

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^2}$$

$$0 \leq \frac{x^2y^2}{x^4+y^2} = \underbrace{\boxed{\frac{y^2}{x^4+y^2}}}_{\leq 1} \cdot x^2 \leq x^2$$

$$\underset{\downarrow \atop 0}{0} \leq \underbrace{\frac{x^2y^2}{x^4+y^2}}_{\downarrow \atop 0} \leq \underset{\downarrow \atop 0}{x^2}$$

$\boxed{\text{Exemple 2}}$

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^4 + y^2}$$

$$\boxed{\frac{x^2 y}{x^4 + y^2}} \leq \frac{1}{2}$$

$$\cdot\, x \to 0$$

$$\boxed{\left|\frac{2ab}{a^2+b^2}\right| \leq 1 \quad ?}$$

$$2|ab| \leq a^2+b^2 \quad ??$$

$$a^2+b^2-2|a|\cdot|b| \geq 0 \quad ??$$

$$(|a|-|b|)^2 \geq 0 \quad ??$$

$\boxed{\text{ESEMPIO 3}}$

$$\lim_{(x,y)\to(0,0)} \frac{x^{100}\cdot y^{100}}{x+y}$$

1) $\boxed{x = 0}$

$$\lim_{y\to 0} 0 = 0 \,.$$

2) $\boxed{y = -x + x^{1000}}$

$$\lim_{x\to 0} \frac{x^{100}\cdot(-x+x^{1000})^{100}}{x - x + x^{1000}}$$

Molto sbagliato

$$\cancel{\frac{x}{x+y}\cdot x^{99}\cdot y^{100}}$$

NO

$$\frac{y^2}{y^2+x^4}$$

$$\lim_{x\to 0} \frac{x^{100}\cdot\left(-x+x^{1000}\right)^{100}}{\cancel{x}-\cancel{x}+x^{1000}} = \lim_{x\to 0}\frac{x^{200}}{x^{1000}} = +\infty$$