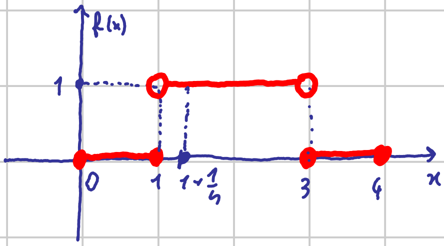
Analisi Matematica (II modulo) - Lez. 3

Titolo nota

9 marzo 2020 (11.00-13.00) - docente: Prof. **Emanuele Callegari** - Università di Roma ***Tor Vergata***

INTEGRALE DI RIEMANN (CALCOLO)

ES. 0 $f(x)=\chi_{(1,3)}(x)$ su $[0,4]$ $\quad \int_0^4 \chi_{(1,3)}(x)\,dx = 2$



(*) $P=\{0,1,3,4\}$ $\quad S(f,P)=\sum_{i=1}^{3}(x_i-x_{i-1})\cdot\sup_{[x_{i-1},x_i]} f(x) =$

$$= 1\cdot 0 + 2\cdot 1 + 1\cdot 0 = 2$$

$$s(f,P)=\sum_{i=1}^{3}(x_i-x_{i-1})\cdot\inf_{[x_{i-1},x_i]} f(x) = 1\cdot 0+2\cdot 0+1\cdot 0 = 0$$

$P_n=\{0,1+\frac{1}{n},3,4\}$

(✱) $$s(f,P_n)=\sum_{i=1}^{3}(x_i-x_{i-1})\cdot\inf_{[x_{i-1},x_i]} f(x) = \left(1+\tfrac{1}{n}\right)\cdot 0+\left(2-\tfrac{1}{n}\right)\cdot 1+1\cdot 0 = 2-\frac{1}{n}$$

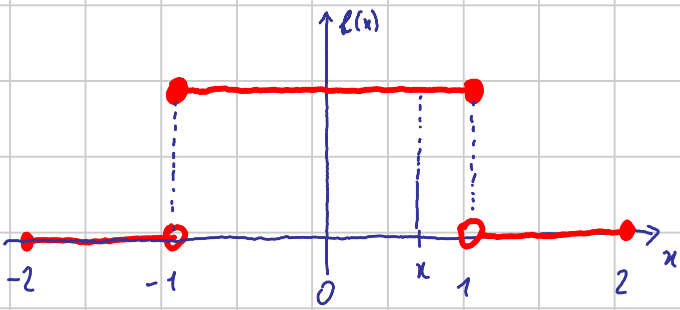
$$\left.\begin{array}{l} (✱) \Rightarrow \int^{-} f \geq 2 \\ (*) \Rightarrow \int^{+} f \leq 2 \end{array}\right\} \Rightarrow \quad 2\leq \int^{-} f \leq \int^{+} f \leq 2 \quad \Rightarrow \quad \int^{-} f = \int^{+} f = 2$$

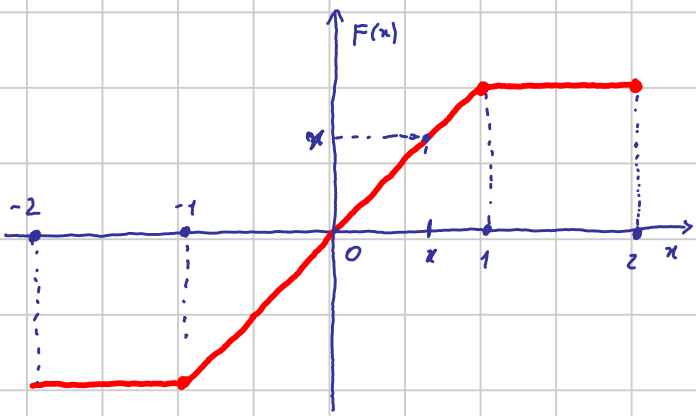
DEF 1 DATI $[a,b]\subset\mathbb{R}$, $x_0\in[a,b]$, $f\in\mathcal{R}([a,b])$ DEFINIAMO

$$F:[a,b]\to\mathbb{R}$$
$$x\mapsto\int_{x_0}^{x} f(t)\,dt$$

FUNZIONE INTEGRALE DI f CON PUNTO BASE x_0.

ES. 1 $[-2,2]$ $\quad f(x)=\chi_{[-1,1]}(x)$ $\quad x_0=0$ $\quad F(x)=\int_0^x \chi_{[-1,1]}(t)\,dt$





$$F(x)=\begin{cases} -1 & \text{se } x\in[-2,-1] \\ x & \text{se } x\in(-1,1) \\ 1 & \text{se } x\in[1,2] \end{cases}$$

TEO. 1 DATI $[a,b]\subset\mathbb{R}$, $x_0\in[a,b]$, $f\in\mathcal{R}([a,b])$ SIA $F(x)=\int_{x_0}^{x} f(t)\,dt$

1) F È LIPSCH. CON COSTANTE $L=\sup_{[a,b]}|f(x)|$

2) (T.F.C.I.) SE f È CONTINUA IN $\bar{x}\in[a,b]$ ALLORA $F'(\bar{x})=f(\bar{x})$.

QUI C'È UN REFUSO: CI VUOLE "$\leq$"

DIMO 1 $\forall x,y\in[a,b]$ $x\neq y$ $\quad \left|\frac{F(y)-F(x)}{y-x}\right|<L$

$\boxed{x<y}$

$$|F(y)-F(x)|=\Big|\underbrace{\int_{x_0}^{y} f(t)\,dt-\int_{x_0}^{x} f(t)\,dt}_{\int_{x_0}^{x}+\int_{x}^{y}}\Big|=\left|\int_x^y f(t)\,dt\right|\leq$$

$$\leq\int_x^y|f(t)|\,dt\leq\int_x^y L\,dt=L\cdot(y-x)=L\cdot|y-x|$$

$$\left|\frac{F(y)-F(x)}{y-x}\right|=\frac{|F(y)-F(x)|}{|y-x|}\leq\frac{L\cdot\cancel{|y-x|}}{\cancel{|y-x|}}=L$$

2
$$\lim_{h\to0}\frac{F(\bar{x}+h)-F(\bar{x})}{h}=\lim_{h\to0}\frac{1}{h}\cdot\left(\int_{x_0}^{\bar{x}+h}f(t)\,dt-\int_{x_0}^{\bar{x}}f(t)\,dt\right)=$$

$$=\lim_{h\to0}\frac{1}{h}\cdot\int_{\bar{x}}^{\bar{x}+h}f(t)\,dt=\lim_{h\to0}\frac{1}{h}\int_{\bar{x}}^{\bar{x}+h}f(\bar{x})+\big(f(t)-f(\bar{x})\big)\,dt=$$

$$=\lim_{h\to0}\left(\frac{1}{h}\int_{\bar{x}}^{\bar{x}+h}f(\bar{x})\,dt+\frac{1}{h}\int_{\bar{x}}^{\bar{x}+h}[f(t)-f(\bar{x})]\,dt\right)=$$

$$=\lim_{h\to0}\left(\frac{1}{\cancel{h}}\cdot f(\bar{x})\cdot\cancel{h}+\frac{1}{h}\int_{\bar{x}}^{\bar{x}+h}f(t)-f(\bar{x})\,dt\right)=\dots=f(\bar{x})$$

$\boxed{h>0}$
$$\frac{1}{h}\cdot\left|\int_{\bar{x}}^{\bar{x}+h}(f(t)-f(\bar{x}))\,dt\right|\leq\frac{1}{h}\cdot\int_{\bar{x}}^{\bar{x}+h}|f(t)-f(\bar{x})|\,dt\overset{h<\delta}{\leq}\frac{1}{h}\cdot\int_{\bar{x}}^{\bar{x}+h}\varepsilon\,dt=\frac{1}{\cancel{h}}\cdot\varepsilon\cancel{h}=\varepsilon$$

$\forall\varepsilon>0\ \exists\delta>0$ t.c. $|t-\bar{x}|<\delta\Rightarrow|f(t)-f(\bar{x})|<\varepsilon$

COR. 1 DATI $[a,b]\subset\mathbb{R}$, $x_0\in[a,b]$ ED $f:[a,b]\to\mathbb{R}$ CONTINUA, DETTA $F(x)=\int_{x_0}^{x}f(t)\,dt$, SI HA CHE F È PRIMITIVA DI f.

DIMO OVVIA (BASTA APPLIC. IL T.1(2) IN OGNI $\bar{x}\in[a,b]$)

COR. 2 DATI $[a,b]\subset\mathbb{R}$ ED $f:[a,b]\to\mathbb{R}$ CONTINUA. SIA G UNA PRIMITIVA DI f.

ALLORA $\int_a^b f(x)\,dx=G(b)-G(a)=[G(x)]_a^b$.

DIMO SIA $F(x)=\int_{x_0}^{x}f(t)\,dt$ $\quad(x_0\in[a,b])$

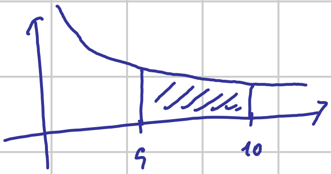
$$F(b)-F(a)=\int_{x_0}^{b}f(x)\,dx-\int_{x_0}^{a}f(x)\,dx=\int_{x_0}^{b}f(x)\,dx+\int_{a}^{x_0}f(x)\,dx=\int_a^b f(x)\,dx$$

$\forall G$ primitiva di f $\exists k$ t.c. $G(x) = F(x) + k$

$$G(b) - G(a) = F(b) \cancel{+k} - (F(a) \cancel{+k}) = F(b) - F(a) = \int_a^b f(x)\,dx$$

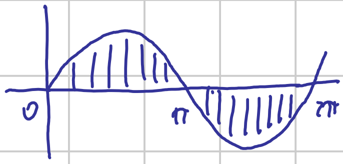
ES. 2

$$\int_5^{10} \frac{1}{x}\,dx = [\ln(x)]_5^{10} = \ln 10 - \ln 5 = \ln 2$$



ES. 3

$$\int_0^{2\pi} \sin x\,dx = [(-\cos x)]_0^{2\pi} = -\cos 2\pi - (-\cos 0) = -1 - (-1) = 0$$



ES. 4

$$\int_0^{\sqrt{3}} \frac{x}{1+x^2}\,dx = \int_0^{\sqrt{3}} \frac{1}{2} \cdot \frac{2x}{1+x^2}\,dx = \left[\frac{1}{2}\ln(1+x^2)\right]_0^{\sqrt{3}} = \frac{1}{2}\ln 4 - \frac{1}{2}\ln 1 = \ln 2$$

ES. 5

$$\int_2^3 \frac{1}{x(x+1)}\,dx = \int_2^3 \frac{x+1-x}{x(x+1)}\,dx = \int_2^3 \left(\frac{x+1}{x(x+1)} - \frac{x}{x(x+1)}\right) dx = \int_2^3 \left(\frac{1}{x} - \frac{1}{x+1}\right) dx =$$

$$= \int_2^3 \frac{1}{x}\,dx - \int_2^3 \frac{1}{x+1}\,dx = [\ln x]_2^3 - [\ln(x+1)]_2^3 = \ln 3 - \ln 2 - \ln 4 + \ln 3 = \ln \frac{9}{8}$$