Analisi Matematica 2 - A.A. 21/22 - Callegari

Lezione 2

8 Marzo 2022

INTEGRALE DI RIEMANN (II)

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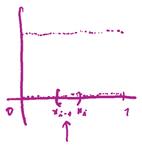
$\boxed{\text{ES.1}}$ $f:[0,1]\to\mathbb{R}$

$$\to f(x)=\begin{cases}1 & \text{SE } x\in\mathbb{Q}\cap[0,1]\\ 0 & \text{ALTRIMENTI}\end{cases}$$



P PART. DI $[0,1]$

$\hookrightarrow = \{x_0, x_1, \dots, x_n\}$ $\quad x_0=0 \quad x_n=1$

$$s(f,P)=\sum_{i=1}^n (x_i-x_{i-1})\overset{0}{m_i}=0$$

$$\underline{\int_{[0,1]}} f = \sup\{0\}=0$$

$$S(f,P)=\sum_{i=1}^n (x_i-x_{i-1})\cdot \overset{1}{M_i}=\sum_{i=1}^n (x_i-x_{i-1})=x_n-x_0=1$$

$\boxed{\text{T.1}}$ DATI $[a,b]\subset\mathbb{R}$ ED $f:[a,b]\to\mathbb{R}$ LIMITATA ALLORA È EQUIV. AFFERMARE CHE:

1) $f\in\mathcal{R}([a,b])$
2) $\forall\varepsilon>0\ \exists P$ PART. DI $[a,b]$ t.c. $S(f,P)-s(f,P)<\varepsilon$

$\boxed{\text{DIM}}$

$(1)\Rightarrow(2)$ $\quad (1)\Rightarrow \underline{\int_{[a,b]}} f=\overline{\int_{[a,b]}} f=\lambda$

$\forall\varepsilon>0$ $\exists P_1$ PART. DI $[a,b]$ t.c. $s(f,P_1)>\lambda-\frac{\varepsilon}{2}$

$\exists P_2$ PART. DI $[a,b]$ t.c. $S(f,P_2)<\lambda+\frac{\varepsilon}{2}$

PRENDO $P=P_1\cup P_2$

$$\lambda-\frac{\varepsilon}{2}<s(f,P_1)\le s(f,P)\le S(f,P)\le S(f,P_2)<\lambda+\frac{\varepsilon}{2}$$

$$\boxed{S(f,P)-s(f,P)<\varepsilon}$$

$(2)\Rightarrow(1)$

$$s(f,P)\le \underline{\int_{[a,b]}} f\le \overline{\int_{[a,b]}} f\le S(f,P)$$

$$0\le \overline{\int_{[a,b]}} f-\underline{\int_{[a,b]}} f\le S(f,P)-s(f,P)<\varepsilon$$

$$\Downarrow$$

$$\overline{\int_{[a,b]}} f-\underline{\int_{[a,b]}} f = 0$$

$\boxed{\text{DEF.1}}$ DATA $f:A\to\mathbb{R}$ ($A\subset\mathbb{R}$) LIMITATA E DATO $B\subset\mathbb{R}$ T.C. $B\cap A\neq\emptyset$

$$\operatorname{osc}(f,B)=\sup\{f(x)\mid x\in B\cap A\}-\inf\{f(x)\mid x\in B\cap A\}$$

$$\text{"}S(f,P)-s(f,P)<\varepsilon\text{"}$$

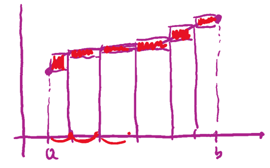
$$S(f,P)-s(f,P)=\sum_{i=1}^n (x_i-x_{i-1})M_i-\sum_{i=1}^n (x_i-x_{i-1})m_i=$$

$$=\sum_{i=1}^n (x_i-x_{i-1})(M_i-m_i)=$$

$$\boxed{=\sum_{i=1}^n (x_i-x_{i-1})\operatorname{osc}(f,[x_{i-1},x_i])}$$

$\boxed{\text{T.2}}$ DATO $[a,b]\subset\mathbb{R}$ E $f\in C([a,b])$ ALLORA $f\in\mathcal{R}([a,b])$

$\boxed{\text{DIM}}$



$\frac{\varepsilon}{b-a}$

PER T. H.-C. f È UNIF. CONT. QUINDI

$\forall\varepsilon>0\ \exists\delta>0$ t.c. $|x-y|<\delta\Rightarrow|f(x)-f(y)|<\frac{\varepsilon}{(b-a)}$

PRENDO $P=\{x_0,x_1,\dots,x_n\}$ t.c. $\forall i=1,\dots,n$ $\ x_i-x_{i-1}<\delta$

$$\boxed{S(f,P)-s(f,P)}=\sum_{i=1}^n (x_i-x_{i-1})\cdot\underbrace{\operatorname{osc}(f,[x_{i-1},x_i])}_{\le\frac{\varepsilon}{b-a}}\le\frac{\varepsilon}{b-a}\cdot\sum(x_i-x_{i-1})=\frac{\varepsilon}{\cancel{b-a}}\cancel{(b-a)}=\varepsilon$$

$$\operatorname{osc}(f,[x_{i-1},x_i])\le\frac{\varepsilon}{b-a}$$

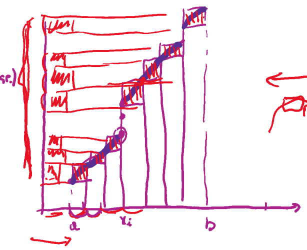
$|f(x)-f(y)|$ $\quad x,y\in[x_{i-1},x_i]$

$$\sup\{|f(x)-f(y)|\ x,y\in[x_{i-1},x_i]\}\le\frac{\varepsilon}{b-a}$$

$\boxed{\text{T.3}}$ DATI $[a,b]\subset\mathbb{R}$ E $f:[a,b]\to\mathbb{R}$ MONOTONA, ALLORA $f\in\mathcal{R}([a,b])$

$\boxed{\text{DIM.}}$

(CASO f CRESC.)



$\forall\varepsilon>0$ PRENDO $P=\{x_0,x_1,\dots,x_n\}$ PART. DI $[a,b]$ TALE CHE $\forall i=1,\dots,n$ $\ x_i-x_{i-1}<\frac{\varepsilon}{f(b)-f(a)}$ $\qquad [x_{i-1},x_i]$

ALLORA

$$S(f,P)-s(f,P)=\sum_{i=1}^n (x_i-x_{i-1})\overset{\le f(x_i)}{M_i}-\sum_{i=1}^n (x_i-x_{i-1})\overset{f(x_{i-1})}{m_i}\le$$

$$\le\sum_{i=1}^n (x_i-x_{i-1})(f(x_i)-f(x_{i-1}))<$$

$$<\sum_{i=1}^n \frac{\varepsilon}{f(b)-f(a)}\cdot(f(x_i)-f(x_{i-1}))=$$

$$=\frac{\varepsilon}{f(b)-f(a)}\cdot\sum_{i=1}^n f(x_i)-f(x_{i-1})=$$

$$=\frac{\varepsilon}{\cancel{f(b)-f(a)}}\cdot\cancel{(f(b)-f(a))}=\varepsilon$$

$\boxed{\text{T.4}}$ DATI $[a,b]\subset\mathbb{R}$ ED $f,g\in\mathcal{R}([a,b])$ ALLORA $\forall\alpha,\beta\in\mathbb{R}$ $\ \alpha f+\beta g\in\mathcal{R}([a,b])$.

E INOLTRE $\int_a^b \alpha f+\beta g\,dx=\alpha\int_a^b f(x)dx+\beta\int_a^b g(x)dx$

$\boxed{\text{DIM}}$ $\boxed{\text{PASSO 1}}$

$\alpha>0$ E $f\in\mathcal{R}([a,b])\Rightarrow\alpha f\in\mathcal{R}([a,b])$ E $\int\alpha f=\alpha\int f$

$\forall P=\{x_0,x_1,\dots,x_n\}$

$$\boxed{s(\alpha f,P)=\alpha\cdot s(f,P)}$$

$$\boxed{s(\alpha f,P)}=\sum_{i=1}^n (x_i-x_{i-1})\cdot\inf\{\alpha f(x)\mid x\in[x_{i-1},x_i]\}=$$

$$=\sum_{i=1}^n (x_i-x_{i-1})\cdot\alpha\cdot\inf\{f(x)\mid x\in[x_{i-1},x_i]\}=$$

$$=\alpha\underbrace{\sum_{i=1}^n (x_i-x_{i-1})\inf\{f(x)\mid x\in[x_{i-1},x_i]\}}_{\boxed{s(f,P)}}=$$

$$\boxed{\underline{\int_{[a,b]}}\alpha f}=\sup\{s(\alpha f,P)\mid P\text{ PART.}\}=$$

$$=\sup\{\alpha\cdot s(f,P)\mid P\text{ PART.}\}=$$

$$=\alpha\sup\{s(f,P)\mid P\text{ PART.}\}=\alpha\cdot\underline{\int_{[a,b]}} f=\alpha\int_a^b f(x)dx$$

$$\boxed{\overline{\int_{[a,b]}}\alpha f}=\alpha\overline{\int_{[a,b]}} f=\alpha\int_a^b f(x)dx$$

$f \qquad -f \qquad P=\{x_0,\dots,x_n\}$

$$\boxed{s(-f,P)}=\sum_{i=1}^n (x_i-x_{i-1})\cdot\inf\{-f(x)\mid x\in[x_{i-1},x_i]\}=$$

$$=\sum_{i=1}^n (x_i-x_{i-1})\cdot(-1)\cdot\sup\{f(x)\mid x\in[x_{i-1},x_i]\}=$$

$$=\boxed{-S(f,P)}$$

$$S(-f,P)=-s(f,P)$$

$$\boxed{\underline{\int_{[a,b]}}-f}=\sup\{s(-f,P)\mid P\text{ PART}\}=$$

$$=\sup\{-S(f,P)\mid P\text{ PART}\}=$$

$$=-\inf\{S(f,P)\mid P\text{ PART}\}=-\overline{\int_{[a,b]}} f=-\int_a^b f(x)dx$$

$$\boxed{\overline{\int_{[a,b]}}-f}=\dots=-\underline{\int_{[a,b]}} f=-\int_a^b f(x)dx$$

$$\int_0^1 x^2$$

$s_n\to\frac{1}{3}$

$$S_n=\frac{1}{3}-\frac{1}{2n}+\dots=\frac{1}{3}+O\left(\frac{1}{n}\right)$$

$f\in C^1([0,1])$ $\quad |f'|\le 5$

STIMARE $|S_n-\int f|$