Analisi Matematica 2 - A.A. 21/22 - Callegari

Lezione 3

9 Marzo 2022

T. 1 DATI $[a,b] \subset \mathbb{R}$ ED $f, g \in \mathcal{R}([a,b])$ ALLORA $f+g \in \mathcal{R}([a,b])$

DIM

(?)

$$\underbrace{\int_a^b f(x)dx}_{} + \underbrace{\int_a^b g(x)dx}_{} \le \underline{\int}_{[a,b]} f+g \le \overline{\int}_{[a,b]} f+g \le \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\underline{\int}_{[a,b]} f \qquad \underline{\int}_{[a,b]} g$$

$\forall \varepsilon > 0$

$$\underline{\int}_{[a,b]} f+g \ge \underline{\int}_{[a,b]} f + \underline{\int}_{[a,b]} g - \varepsilon \quad (?)$$

(?) $\forall \varepsilon > 0 \ \exists \mathcal{P}$ PART. DI $[a,b]$ t.c.

$$s(f+g, \mathcal{P}) > \underline{\int}_{[a,b]} f + \underline{\int}_{[a,b]} g - \varepsilon \quad (?)$$

$$\forall \mathcal{P} = \{x_0, \dots, x_n\} \ s(f+g, \mathcal{P}) \ge s(f,\mathcal{P}) + s(g,\mathcal{P})$$

$$\sum_{i=1}^{n} (x_i - x_{i-1}) \cdot \mathrm{INF}\{f(x)+g(x) \mid x \in [x_{i-1}, x_i]\} \ge$$

$$\ge \sum_{i=1}^{n} (x_i - x_{i-1}) \Big(\underbrace{\mathrm{INF}\{f(x) \mid x \in [x_{i-1}, x_i]\}}_{\lambda} + \underbrace{\mathrm{INF}\{g(x) \mid x \in [x_{i-1}, x_i]\}}_{\mu} \Big) =$$

$$= \sum_{i=1}^{n} (x_i - x_{i-1}) \, \mathrm{INF}\{f(x) \mid x \in I_i\} + \sum_{i=1}^{n} (x_i - x_{i-1}) \, \mathrm{INF}\{g(x) \mid x \in I_i\} =$$

$$= s(f, \mathcal{P}) + s(g, \mathcal{P})$$

PER DEF. $\exists \mathcal{P}_1$ t.c. $s(f, \mathcal{P}_1) > \underline{\int}_{[a,b]} f - \frac{\varepsilon}{2}$

$\boxed{\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2}$

$\exists \mathcal{P}_2$ t.c. $s(g, \mathcal{P}_2) > \underline{\int}_{[a,b]} g - \frac{\varepsilon}{2}$

$s(f, \mathcal{P}) \qquad s(g, \mathcal{P})$

$$s(f+g, \mathcal{P}) \ge \boxed{s(f,\mathcal{P}) + s(g,\mathcal{P}) > \underline{\int}_{[a,b]} f + \underline{\int}_{[a,b]} g - \varepsilon}$$

$$\forall \varepsilon > 0 \quad \exists \mathcal{P} \text{ t.c. } s(f+g, \mathcal{P}) > \underline{\int}_{[a,b]} f + \underline{\int}_{[a,b]} g - \varepsilon$$

$$\forall \varepsilon > 0 \quad \underline{\int}_{[a,b]} f+g > \underline{\int}_{[a,b]} f + \underline{\int}_{[a,b]} g - \varepsilon$$

$$\boxed{\underline{\int}_{[a,b]} f+g \ge \underline{\int}_{[a,b]} f + \underline{\int}_{[a,b]} g}$$

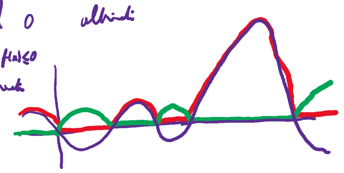
$$\boxed{\overline{\int}_{[a,b]} f+g \le \overline{\int}_{[a,b]} f + \overline{\int}_{[a,b]} g}$$

$$\int_a^b f(x)dx + \int_a^b g(x)dx = \underline{\int} f + \underline{\int} g \le \underline{\int} f+g \le \overline{\int} f+g \le \overline{\int} f + \overline{\int} g = \int_a^b f(x)dx + \int_a^b g(x)dx$$

DEF. 1 DATA $f: [a,b] \to \mathbb{R}$ DEFINIAMO $f^+: [a,b] \to \mathbb{R}$ E $f^-: [a,b] \to \mathbb{R}$

$$f^+(x) = \begin{cases} f(x) & \text{SE } f(x) \ge 0 \\ 0 & \text{altrimenti} \end{cases}$$

$$f^-(x) = \begin{cases} -f(x) & \text{SE } f(x) \le 0 \\ 0 & \text{altrimenti} \end{cases}$$



$$|f(x)| = f^+(x) + f^-(x)$$

$$f(x) = f^+(x) - f^-(x) \qquad f^- = -(f - f^+)$$

T. 2 DATI $[a,b] \subset \mathbb{R}$ $f, g \in \mathcal{R}([a,b])$ ALLORA

$$f^+, f^-, |f|, \mathrm{MAX}(f,g), \mathrm{MIN}(f,g) \in \mathcal{R}([a,b])$$

DIM $\boxed{f^+}$

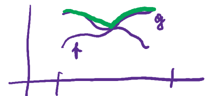
DEVO DIM.

$$\forall \varepsilon > 0 \ \exists \mathcal{P} \text{ PART. DI } [a,b] \text{ t.c. } \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot \mathrm{osc}(f^+, [x_{i-1}, x_i]) < \varepsilon \quad \text{(OVVIO)}$$

OSS. $\mathrm{osc}(f, B) \ge \mathrm{osc}(f^+, B)$ (OVVIO)

$f^- = -(f - f^+)$

$|f| = f^+ + f^-$



$$\mathrm{MAX}(f,g) = f + (g-f)^+$$

$$\mathrm{MIN}(f,g) = -\mathrm{MAX}(-f, -g)$$

T. 3 DATI $[a,b] \subset \mathbb{R}$ E $f, g \in \mathcal{R}([a,b])$, SE $\forall x \in [a,b]$ $f(x) \ge g(x)$ ALLORA

$$\int_a^b f(x)dx \ge \int_a^b g(x)dx$$

DIM BASTA MOSTRARE CHE $\forall \mathcal{P}$ PART. DI $[a,b]$

$$s(f, \mathcal{P}) \ge s(g, \mathcal{P}) \ ?$$

$$\sum_{i=1}^{n} (x_i - x_{i-1}) \, \mathrm{INF}\{f(x) \mid x \in [x_{i-1}, x_i]\} \ge \sum_{i=1}^{n} (x_i - x_{i-1}) \, \mathrm{INF}\{g(x) \mid x \in [x_{i-1}, x_i]\}$$

T. 4 DATI $[a,b] \subset \mathbb{R}$, $f \in \mathcal{R}([a,b])$ ALLORA

$$\left| \int_a^b f(x)dx \right| \le \int_a^b |f(x)| dx$$

DIM

(?)

$$-\int_a^b |f(x)| dx \le \int_a^b f(x)dx \le \int_a^b |f(x)| dx$$

$f(x) \le |f(x)| \quad \forall x$

$$\int_a^b f(x)dx \ge -\int_a^b |f(x)| dx \quad (?)$$

$$\int_a^b f(x)dx \ge \int_a^b -|f(x)| dx \quad (?)$$

SI PERCHÉ $f(x) \ge -|f(x)|$ (OVVIO)

T. 5 DATI $[a,b] \subset \mathbb{R}$ E $f \in \mathcal{R}([a,b])$ E $c \in (a,b)$ ALLORA $f \in \mathcal{R}([a,c])$, $f \in \mathcal{R}([c,b])$ E

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

DIM

$\forall \varepsilon > 0$ SIA $\mathcal{P} = \{x_0, \dots, x_n\}$ T.C.

$$\sum_{i=1}^{n} (x_i - x_{i-1}) \, \mathrm{osc}(f, I_i) < \varepsilon \qquad I_i = [x_{i-1}, x_i]$$

WLOG $c \in \mathcal{P}$ ALTRIMENTI POSSO COMUNQUE AGGIUNGERLO E IL 1° MEMBRO DIMINUISCE

$$\mathcal{P} = \{\underbrace{x_0, x_1, \dots,}_{\mathcal{P}_1} \overbrace{c, \dots, x_n}^{\mathcal{P}_2}\} \qquad c = x_k$$

$$\sum_{i=1}^{k} (x_i - x_{i-1}) \, \mathrm{osc}(f, I_i) + \sum_{i=k+1}^{n} (x_i - x_{i-1}) \, \mathrm{osc}(f, I_i) =$$

$\mathcal{P}_1$ $[a,c]$ $\qquad$ $\mathcal{P}_2$ $[c,b]$

$$\int_a^c f(x)dx + \int_c^b f(x)dx \le \int_a^b f(x)dx \le \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\boxed{\forall \varepsilon > 0 \quad \int_a^b f(x)dx < \int_a^c f(x)dx + \int_c^b f(x)dx + \varepsilon}$$

$$\forall \varepsilon > 0 \quad \overline{\int}_{[a,b]} f < \int_{[a} \dots$$

$$\forall \varepsilon > 0 \quad \overline{\int}_{[a,b]} f < \boxed{\overline{\int}_{[a,c]} f + \frac{\varepsilon}{2} + \overline{\int}_{[c,b]} f + \frac{\varepsilon}{2}} \ ?$$

$\forall \varepsilon > 0 \ \exists \mathcal{P}$ DI $[a,b]$ t.c.

$$S(f, \mathcal{P}) <$$

PRENDO $\mathcal{P}_1 = \{x_0, x_1, \dots, x_k\}$ PART. DI $[a,c]$ T.C.

$$S(f, \mathcal{P}_1) < \overline{\int}_{[a,c]} f + \frac{\varepsilon}{2}$$

PRENDO $\mathcal{P}_2 = \{x_k, \dots, x_n\}$ PART. DI $[c,b]$ t.c.

$$S(f, \mathcal{P}_2) < \overline{\int}_{[c,b]} f + \frac{\varepsilon}{2}$$

$\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$

$$\boxed{S(f, \mathcal{P})} = S(f, \mathcal{P}_1) + S(f, \mathcal{P}_2) \le \square + \square$$