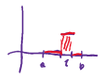
Analisi Matematica 2 - A.A. 21/22 - Callegari

Lezione 4

11 Marzo 2022

→ 1) MISURA P.J. (DEF. ESIS)
2) FUNZIONE INTEGRALE (DEF NUOVA)
3) T. F.C.I.
4) CONSEGUENZE
5) ESEMPI $\left(\int \frac{1}{\sin x}, \int \frac{x}{1+x^2}\right)$
6) INT. PER PARTI
7) ESEMPI $\left(\int x e^x \cos x\right)$
8) INT. PER SOST.
9) ESEMPI $\left(\int \frac{1}{\sin x}, \int \frac{1}{\cos x}\right)$

ES. 0



$m(I) = \int \chi_I(x)$

$I_1 \cap I_2 = \emptyset$

$m(I_1) + m(I_2) = \int_a^b \chi_{I_1}(x) + \int_a^b \chi_{I_2}(x) =$

$= \int_a^b \chi_{I_1}(x) + \chi_{I_2}(x)\,dx =$

$= \int_a^b \chi_{I_1 \cup I_2}(x)\,dx =$

$= m(I_1 \cup I_2)$ (*)

→ $(I_i)_{i\in\mathbb{N}}$ DISG.

$m\left(\bigcup_{i\in\mathbb{N}} I_i\right) = \sum_{i=1}^{+\infty} m(I_i)$

DEF 1 DATO $[a,b] \subset \mathbb{R}$ E DATO $A \subset [a,b]$ DEFINIAMO

$m(A) = \int_a^b \chi_A(x)\,dx$ SE $\chi_A(x) \in \mathcal{R}([a,b])$

IN TAL CASO SI DICE CHE A È MIS. SECONDO P.J.

OSS 1 IL CALCOLO (*) DIMOSTRA CHE SE A, B DISC. E MISUR. ALLORA
1) $A \cup B$ È MIS.
2) $m(A \cup B) = m(A) + m(B)$

OSS. 2 VALE ANCHE LA PROP.: SE $A \subset B$, ENTRAMBI MIS. ALLORA

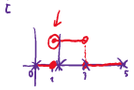
$m(A) \le m(B)$

$\int_a^b \chi_A(x)\,dx \le \int_a^b \chi_B(x)\,dx$

$A \subset B \Rightarrow \chi_A(x) \le \chi_B(x)$

ES. 1 DATO $(s,t) \subset [0,5]$

$m((s,t)) = 2$



$[s,t]$

$P = \{0, 1, 3, 5\}$

$s(\chi(I), P) = 0 + 2 + 0 = 2$

$S(\chi(I), P) = \cdots = 2$

$P_n = \{0, 1+\frac{1}{n}, 3, 5\}$

$s(\chi(I), P_n) = 0 + \left(2 - \frac{1}{n}\right) + 0$

$2 - \frac{1}{n} \le m(I) \le 2 \quad \forall n$ in fact.

②

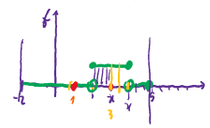
DEF 2 DATI $[a,b] \subset \mathbb{R}$, $x_0 \in [a,b]$, $f \in \mathcal{R}([a,b])$

DEFINIAMO

$F : [a,b] \to \mathbb{R}$, $x \mapsto \int_{x_0}^x f(t)\,dt$

FUNZIONE INT. DI f CON P.I. IN x_0

ES. 2 $[-2,5]$ $f(x) = \chi_{[0,4]}(x)$ $x_0 = 1$, $x_0 = 3$



T. 1 DATI $[a,b] \subset \mathbb{R}$, $x_0 \in [a,b]$, $f \in \mathcal{R}([a,b])$

SIA $F(x) = \int_{x_0}^x f(t)\,dt$ $\forall x \in [a,b]$ ALLORA:

1) F È LIPSCH. CON COST. $L = \sup\{|f(t)| \mid t \in [a,b]\}$

(T.F.C.I.2) SE IN $\bar{x} \in (a,b)$ f È CONT. ALLORA F È DERIVABILE IN $\bar{x}$ E $F'(\bar{x}) = f(\bar{x})$

DIM 1) $\forall x, y \in [a,b]$ $|F(x) - F(y)| \le L \cdot |x - y|$?

$x > y$ (il caso $x < y$ è analogo)

$|F(x) - F(y)| = \left|\int_{x_0}^x - \int_{x_0}^y\right| = \left|\int_y^x f(t)\,dt\right| \le$

$\le \int_y^x |f(t)|\,dt \le \int_y^x L\,dt =$

$= L \int_y^x 1\,dt = L(x - y) = L|x - y|$

$|F(x) - F(y)| \le L \cdot |x - y|$

② $F'(\bar{x}) = \lim_{h\to 0} \frac{F(\bar{x}+h) - F(\bar{x})}{h} =$

$= \lim_{h\to 0} \frac{\int_{\bar{x}}^{\bar{x}+h} f(t)\,dt}{h} =$

$= \lim_{h\to 0} \frac{1}{h}\left(\int_{\bar{x}}^{\bar{x}+h} f(t) - f(\bar{x}) + f(\bar{x})\,dt\right) =$

$= \lim_{h\to 0}\left(\frac{1}{h}\cdot\int_{\bar{x}}^{\bar{x}+h} f(\bar{x})\,dt + \frac{1}{h}\int_{\bar{x}}^{\bar{x}+h}(f(t) - f(\bar{x}))\,dt\right) =$

$= \lim_{h\to 0}\left(f(\bar{x}) + \frac{1}{h}\int_{\bar{x}}^{\bar{x}+h}(f(t) - f(\bar{x}))\,dt\right) = \cdots = f(\bar{x})$

$\forall \varepsilon > 0\ \exists \delta > 0$ t.c. $|t - \bar{x}| < \delta \Rightarrow |f(t) - f(\bar{x})| < \varepsilon$

$h < \delta$

$|f(t) - f(\bar{x})| < \varepsilon$

$|\Phi| \le \frac{1}{h}\cdot\int_{\bar{x}}^{\bar{x}+h} |f(t) - f(\bar{x})|\,dt \le \frac{1}{h}\int_{\bar{x}}^{\bar{x}+h} \varepsilon\,dt = \frac{1}{h}\cdot\varepsilon\cdot h = \varepsilon$

$h < \delta$

COR. 1 DATO $[a,b] \subset \mathbb{R}$ E $f \in \mathcal{C}([a,b])$ ALLORA

$F(x) = \int_{x_0}^x f(t)\,dt$ $\forall x_0 \in [a,b]$

$F'(x) = f(x)$ $\forall x \in (a,b)$

COR. 2 DATA f [... COME PRIMA] TALE $G : (a,b) \to \mathbb{R}$ PRIMITIVA DI f. ALLORA

$\int_a^b f(x)\,dx = G(b) - G(a)$

DIM. PRENDO $F(x) = \int_a^x f(t)\,dt$ E

$F'(x) = f(x)$

$F(b) - F(a) = \int_a^b f(t)\,dt - 0$

$G(x) = F(x) + C$

$G(b) - G(a) = F(b) + C - (F(a) + C) =$

$= \int_a^b f(t)\,dt$

PROP. 1 DA $G : [a,b] \to \mathbb{R}$ SE $F, G : [a,b] \to \mathbb{R}$ SONO TALI CHE

$\forall x \in [a,b)$ $G'(x); f(x) = F'(x)$

ALLORA $\exists c \in \mathbb{R}$ t.c. $\forall x \in [a,b]$

$F(x) = G(x) + c$

DIM

SE $G'(x) = F'(x)$ $\exists c \in \mathbb{R}$ t.c. $\forall x \in [a,b]$

$G(x) - F(x) = c$

$\Downarrow$

$(G(x) - F(x))' = 0$

$H(x)$

$H'(x) = 0 \quad \forall x \in (a,b)$

SE P.A.

$H(x_1) - H(x_2) \neq 0$

$x_1 - x_2$

$\Downarrow$

$H'(x_0) \neq 0$ ↯ T.S.

$$\int_0^1 e^x\,dx = \left[e^x\right]_0^1 = e^1 - e^0 = e - 1$$

T. 2 DATI $f, g \in \mathcal{C}^1([a,b])$

$$\int_a^b f(x)\,g'(x)\,dx = \left[f(x)\cdot g(x)\right]_a^b - \int_a^b f'(x)\,g(x)\,dx$$

DIM

$(f(x)\cdot g(x))' = f'(x)\cdot g(x) + f(x)\cdot g'(x)$

$f(x)\cdot g'(x) = (f(x)\cdot g(x))' - f'(x)\cdot g(x)$

$\int_a^b f(x)\,g'(x)\,dx = \int_a^b (f(x)\,g(x))'\,dx - \int_a^b f'(x)\,g(x)\,dx$

$\left[f(x)\cdot g(x)\right]_a^b$

$$\int_0^1 \frac{1}{(1+x^2)^2}\,dx = \int_0^1 \frac{1+x^2-x^2}{(1+x^2)^2}\,dx =$$

$$= \int_0^1 \left(\frac{1}{1+x^2} - \frac{x^2}{(1+x^2)^2}\right)dx$$

$\int_0^Y \frac{1}{1+x^2}\,dx - \int_0^1 \frac{x^2}{(1+x^2)^2}\,dx$

$$\int_0^1 \frac{x^2}{(1+x^2)^2}\,dx =$$

$$= \int_0^1 1 + \frac{x}{2}\cdot\left(\frac{+2x}{(1+x^2)^2}\right)dx =$$

$$= \int_0^1 1 + \frac{x}{2}\cdot\left(-\frac{1}{1+x^2}\right)'dx =$$

$$= \left[-\frac{x}{2(1+x^2)}\right]_0^1 + \frac{1}{2}\int_0^1 \frac{1}{1+x^2}\,dx$$

$\left(-\frac{1}{1+x^2}\right)' = +\frac{1}{(1+x^2)^2}\cdot 2x$