Analisi Matematica 2 - A.A. 21/22 - Callegari

Lezione 5

14 Marzo 2022

2) INT. PER PARTI (REGOLA) ←

3) ESEMPI
A) $\int_0^1 x e^x \, dx$ ←
B) $\int_{-\pi}^{\pi} x^2 \cos 2x \, dx$ ←
C) $\int_0^1 \arctan x \, dx$
D) $\int_1^e x \ln x \, dx$
E) $\int_0^1 (x^3+5x^2) \ln(1+x^2) \, dx$
F) $\int_0^1 e^{2x} \sin 4x \, dx$

4) INT. PER SOSTITUZIONE (REGOLA)

5) ESEMPI
A) $\int_4^9 \frac{1}{x+\sqrt{x}} \, dx$
B) $\int_0^1 \sqrt{4-x^2} \, dx$
C) $\int_{\frac{25}{24}}^{\frac{13}{5}} \sqrt{x^2-1} \, dx$
D) $\int_0^4 \frac{1}{\sqrt{9x^2+25}} \, dx$

INT. DI RIEMANN (TECNICHE DI CALCOLO)

ES. P. PARTI

$$\int_0^1 x e^x \, dx = \int_0^1 x (e^x)' \, dx = [x \cdot e^x]_0^1 - \int_0^1 (x)' e^x \, dx =$$

$$\cancel{= \int_0^1 \left(\frac{x^2}{2}\right)' e^x \, dx = \left[\frac{x^2}{2} \cdot e^x\right]_0^1 - \int_0^1 \frac{x^2}{2} \cdot e^x \, dx}$$

$$= e - \int_0^1 e^x \, dx = e - [e^x]_0^1 = e - (e-1) = 1$$

$$\int_a^b P(x) e^{ax+b} = \dots$$

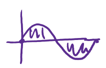
$$\int_2^3 (x^3+2x) e^{5x} \, dx = \int_2^3 (x^3+2x)\left(\frac{1}{5} e^{5x}\right)' dx =$$

$$= \left[\frac{1}{5} e^{5x} (x^3+2x)\right]_2^3 - \int_2^3 (3x^2+2) \cdot \frac{1}{5} e^{5x} \, dx =$$

ES. B

$$\int_{-\pi}^{\pi} x^2 \cos 2x \, dx = \int_{-\pi}^{\pi} x^2 \cdot \left(\frac{1}{2} \sin 2x\right)' dx =$$

$$= \left[x^2 \cdot \frac{1}{2} \sin 2x\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x \cdot \frac{1}{2} \sin 2x \, dx =$$



$$= 0 + \int_{-\pi}^{\pi} x \cdot \left(\frac{1}{2} \cos 2x\right)' dx =$$

$$= \left[x \cdot \frac{1}{2} \cos 2x\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{1}{2} \cos 2x \, dx =$$

$$= \pi - 0 = \pi$$

$$\int_1^e (x^2+5) \ln x \, dx = \int_1^e \left(\frac{x^3}{3}+5x\right)' \cdot \ln x \, dx =$$

$$= \left[\left(\frac{x^3}{3}+5x\right) \ln x\right]_1^e - \int_1^e \left(\frac{x^3}{3}+5x\right) \cdot \frac{1}{x} \, dx =$$

$$= \frac{e^3}{3} + 5e - \int_1^e \left(\frac{x^2}{3} + 5\right) dx$$

$$= \frac{e^3}{3} + 5e - \left[\frac{x^3}{9} + 5x\right]_1^e =$$

$$= \frac{e^3}{3} + \cancel{5e} - \left(\frac{e^3}{9} + \cancel{5e} - \frac{1}{9} - 5\right) =$$

$$= \boxed{\frac{2}{9} e^3 + \frac{16}{5}}$$

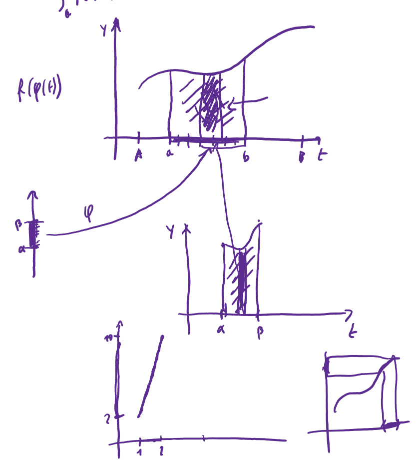
$$\int_0^1 1 \cdot \arctan x \, dx = \int_0^1 (x)' \cdot \arctan x \, dx =$$

$$= [x \arctan x]_0^1 - \frac{1}{2} \int_0^1 \frac{2x}{1+x^2} \, dx =$$

$$= \frac{\pi}{4} - \frac{1}{2} \left[\ln(1+x^2)\right]_0^1 =$$

$$= \frac{\pi}{4} - \frac{1}{2} \ln 2$$

$$\int_a^b f(x) \, dx$$



T. 2 DATI $[A,B]$, $[a,b]$, $[\alpha,\beta] \subset \mathbb{R}$ t.c. $[a,b] \subset (A,B)$

$f \in C([A,B])$

$\varphi \in C^1([\alpha,\beta])$ $\quad \varphi: [\alpha,\beta] \to (A,B)$ t.c. $\varphi(\alpha) = a$, $\varphi(\beta) = b$

ALLORA

$$\int_a^b f(x) \, dx = \int_\alpha^\beta f(\varphi(t)) \varphi'(t) \, dt$$

DIM

PRENDO $x_0 \in [A,B]$ E PONGO $F(x) = \int_{x_0}^x f(u) \, du$

$$G(t) = F(\varphi(t))$$

$$G'(t) = F'(\varphi(t)) \cdot \varphi'(t) = f(\varphi(t)) \varphi'(t)$$

$$\boxed{\int_\alpha^\beta f(\varphi(t)) \varphi'(t) \, dt} = [F(\varphi(t))]_\alpha^\beta =$$

$$= F(\varphi(\beta)) - F(\varphi(\alpha)) =$$

$$= F(b) - F(a) = \boxed{\int_a^b f(x) \, dx}$$

$$\int_a^b f(x) \, dx = \int_\alpha^\beta f(\varphi(t)) \varphi'(t) \, dt$$

$\varphi: [2,3] \to [4,9]$
$t \mapsto t^2$

ES. 1

$$\boxed{\int_4^9 \frac{1}{x+\sqrt{x}} \, dx} =$$

$$= \int_2^3 \frac{1}{t^2+\sqrt{t^2}} \cdot 2t \, dt =$$

$$= \int_2^3 \frac{1}{t^2+t} \cdot 2t \, dt =$$

$$= \boxed{\int_2^3 \frac{2}{t+1} \, dt} =$$

$$= [2 \ln(t+1)]_2^3 = 2 \ln 4 - 2 \ln 3 =$$

$$= \boxed{2 \ln \frac{4}{3}}$$

$$\int_4^9 \frac{1}{t+\sqrt{t}} \, dt = \int_4^9 \frac{2}{\sqrt{t}+1} \left(\frac{1}{2\sqrt{t}}\right) dt =$$

$$= \int_4^9 \frac{2}{\sqrt{t}+1} \cdot (\overbrace{\sqrt{t}+1}^{\varphi(t)})' \, dt =$$

$$= \int_3^4 \frac{2}{x} \, dx = [2 \ln x]_3^4 = \boxed{2 \ln \frac{4}{3}}$$

ES. B

$$\int_0^1 \sqrt{4-x^2} \, dx = \qquad \boxed{\varphi(t) = 2 \sin t}$$

$$= \int_0^{\frac{\pi}{6}} \sqrt{4-4\sin^2 t} \cdot 2\cos t \, dt =$$

$$= \int_0^{\frac{\pi}{6}} 2 \overbrace{\sqrt{1-\sin^2 t}}^{\cos t} \cdot 2\cos t \, dt =$$

$$= \int_0^{\frac{\pi}{6}} 4 \cos^2 t \, dt =$$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$

$$= \int_0^{\frac{\pi}{6}} 2\cos 2t + 2 \, dt =$$

$$= [\sin 2t + 2t]_0^{\frac{\pi}{6}} = \boxed{\frac{\sqrt{3}}{2} + \frac{\pi}{3}}$$

$$\boxed{\int_{\frac{25}{24}}^{\frac{12}{5}} \sqrt{x^2-1} \, dx} =$$

sh t
ch t
$x^2 - y^2 = 1$

$ch^2 t - sh^2 t = 1$

$(sh\, t)' = ch\, t$
$(ch\, t)' = sh\, t$

$$= \int \sqrt{ch^2 t - 1} \cdot sh\, t \, dt =$$

$$= \int \sqrt{sh^2 t} \cdot sh\, t \, dt = \int sh^2 t \, dt$$

$$\int F\left(x, \sqrt{ax^2+bx+c}\right)$$

$\sqrt{1-x^2}$ — cos t, sin t

$\sqrt{x^2-1}$ — ch t

$\sqrt{x^2+1}$ — sh t

$$\boxed{sh\, t = \frac{e^t - e^{-t}}{2} \qquad ch\, t = \frac{e^t + e^{-t}}{2}}$$

$$(sh\, t)' = \left(\frac{e^t - e^{-t}}{2}\right)' = \frac{e^t + e^{-t}}{2} = ch\, t$$

$$(ch\, t)' = \left(\frac{e^t + e^{-t}}{2}\right)' = \frac{e^t - e^{-t}}{2} = sh\, t$$

$$(ch\, t)^2 - (sh\, t)^2 = \left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2 =$$

$$= \frac{\cancel{e^{2t}} + \cancel{e^{-2t}} + 2}{4} - \frac{\cancel{e^{2t}} + \cancel{e^{-2t}} - 2}{4} =$$

$$= \frac{4}{4} = 1$$

$x = ch\, t$

$$\int_{\frac{25}{24}}^{\frac{12}{5}} \sqrt{x^2-1} \, dx = \int_{\ln \frac{4}{3}}^{\ln 5} \sqrt{ch^2 t - 1} \; sh\, t \, dt = \int_{\ln \frac{4}{3}}^{\ln 5} sh^2 t \, dt = \int_{\ln \frac{4}{3}}^{\ln 5} \left(\frac{e^t - e^{-t}}{2}\right)^2 dt =$$

$$= \int_{\ln \frac{4}{3}}^{\ln 5} \frac{1}{4}\left(e^{2t} + e^{-2t}\right) - \frac{1}{2} \, dt =$$

$$= \left[\frac{1}{8} e^{2t} - \frac{1}{8} e^{-2t} - \frac{1}{2} t\right]_{\ln \frac{4}{3}}^{\ln 5} =$$

$$= \frac{1}{8} 25 - \frac{1}{8} \cdot \frac{1}{25} - \frac{1}{2} \ln 5 -$$

$$- \frac{1}{8} \frac{16}{9} + \frac{1}{8} \frac{9}{16} + \frac{1}{2} \ln \frac{4}{3} =$$

$$= \dots\dots$$

$ch\, t = \frac{25}{24} \iff e^t + e^{-t} = \frac{25}{12}$

$(e^t)^2 + 1 = \frac{25}{12}(e^t) \qquad z = e^t$

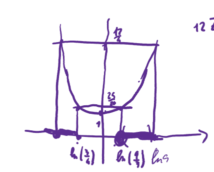
$12 z^2 - 25 z + 12 = 0$

$$z = \frac{25 \pm \sqrt{25^2 - 24^2}}{24} =$$

$$= \frac{25 \pm 7}{24}$$

$$e^t = \begin{cases} \frac{18}{24} = \frac{3}{4} \\ \frac{32}{24} = \frac{4}{3} \end{cases}$$

$t = \ln \frac{3}{4}$
$t = \ln \frac{4}{3}$



$ch\, t = \frac{13}{5} =$

$e^t + e^{-t} = \frac{26}{5}$

$5(e^t)^2 - 26(e^t) + 5 = 0$

$5(e^t)^2 - 25(e^t) - e^t + 5 = 0$

$5e^t(e^t - 5) - (e^t - 5) = 0$

$(e^t - 5)(5e^t - 1) = 0$

$e^t = 5 \qquad t = \ln 5$
$e^t = \frac{1}{5} \qquad t = \ln \frac{1}{5}$