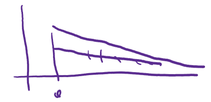
Analisi Matematica 2 - A.A. 21/22 - Callegari

Lezione 8

21 Marzo 2022

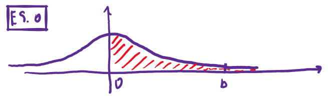


0) ES. INTRODUTTIVO
1) DEF.
2) ESEMPI
3) $\mathcal{R}^*([a,b))$ È SP. VETT.
4) SPOSTAMENTO DEL PUNTO BASE
5) PIÙ CATTIVERIE

6) POSITIVITÀ INTEGRANDA
7) CR. CONF.
8) CR. CONF. AS.
9) CASI NOTEVOLI

INTEGRALE IMPROPRIO
(DEFINIZIONI ED ESEMPI)

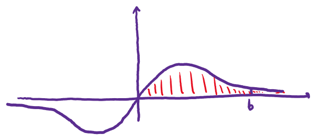
ES. 0



$f(x)=\frac{1}{1+x^2}$

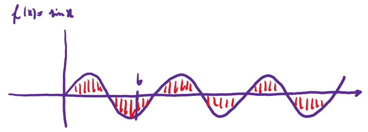
$$\int_0^{+\infty} f(x)\,dx = \lim_{b\to+\infty}\left(\int_0^b \frac{1}{1+x^2}dx\right) = \lim_{b\to+\infty}\left[\arctan x\right]_0^b = \lim_{b\to+\infty}\arctan b = \frac{\pi}{2}$$

ES. 1 $f(x)=\frac{2x}{1+x^2}$



$$\lim_{b\to+\infty}\left(\int_0^b \frac{2x}{1+x^2}dx\right) = \lim_{b\to+\infty}\left[\ln(1+x^2)\right]_0^b =$$
$$= \lim_{b\to+\infty} \ln(1+b^2) = +\infty$$

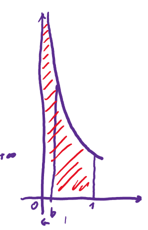
ES. 2 $f(x)=\sin x$



$$\lim_{b\to+\infty}\left(\int_0^b \sin x\,dx\right) = \lim_{b\to+\infty}\left[\cos x\right]_0^b = \lim_{b\to+\infty}(1-\cos b) =$$

ES. 3 $f(x)=\frac{1}{x^2}$

$$\lim_{b\to0^+}\int_b^1 \frac{1}{x^2}dx =$$
$$= \lim_{b\to0^+}\left[-\frac{1}{x}\right]_b^1 = \lim_{b\to0^+}\left(-1+\frac{1}{b}\right) = +\infty$$



ES. 4 $f(x)=\frac{1}{\sqrt{x}}$

$$\lim_{b\to0^+} 2\int_b^1 \frac{1}{2\sqrt{x}}dx = \lim_{b\to0^+} 2\cdot\left[\sqrt{x}\right]_b^1 = \lim_{b\to0^+}\left(2\sqrt{1}-2\sqrt{b}\right) = 2$$

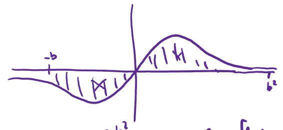
DEF. 1 DATA $f:[a,b)\to\mathbb{R}$, CON $b\in\mathbb{R}\cup\{+\infty\}$ t.c. $f\in\mathcal{R}([a,c])$ $\forall c\in[a,b)$, ALLORA:

1) SE $\lim_{c\to b^-}\int_a^c f(x)dx = \lambda\in\mathbb{R}$ DIREMO CHE $\int_a^b f(x)dx$ CONVERGE A λ
2) SE $\lim_{c\to b^-}\int_a^c f(x)dx = +\infty$ $(-\infty)$ DIREMO CHE $\int_a^b f(x)dx$ DIVERGE A $+\infty$ $(-\infty)$
3) " " " " " NON ESISTE " " . . . . È INDET.

ES. CATTIVO

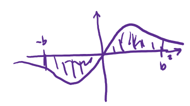
$f(x)=\frac{2x}{1+x^2}$

$$\int_{-\infty}^{+\infty}\frac{2x}{1+x^2}dx \overset{\text{NO}}{=} \lim_{b\to+\infty}\int_{-b}^{b}\frac{2x}{1+x^2}dx =$$
$$= \lim_{b\to+\infty}\left(\left[\ln(1+x^2)\right]_{-b}^{b}\right) = \lim_{b\to+\infty} 0 = 0$$



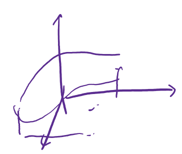
$$\lim_{b\to+\infty}\int_{-b}^{b^2}\frac{2x}{1+x^2}dx = \lim_{b\to+\infty}\left[\ln(1+x^2)\right]_{-b}^{b^2} =$$
$$= \lim_{b\to+\infty}\left(\ln(1+b^4)-\ln(1+b^2)\right) =$$
$$= \lim_{b\to+\infty}\ln\left(\frac{1+b^4}{1+b^2}\right) = +\infty$$

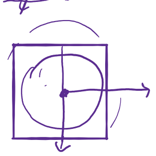
$$\lim_{b\to+\infty}\int_{-b}^{2b}\frac{2x}{1+x^2}dx = \lim_{b\to+\infty}\ln(1+4b^2)-\ln(1+b^2) =$$
$$\ln\left(\frac{1+4b^2}{1+b^2}\right) = \ln 4$$
$$= \lim_{b\to+\infty}$$



$\frac{2x}{1+x^4}$

$$\int_0^{+\infty}\frac{2x}{1+x^4}dx = \lim_{b\to+\infty}\int_0^b\frac{2x}{1+x^4}dx =$$
$$= \lim_{b\to+\infty}\int_0^b\frac{1}{1+(x^2)^2}\cdot(x^2)'dx =$$
$$= \lim_{b\to+\infty}\left[\arctan x^2\right]_0^b = \frac{\pi}{2}$$





DEF. 2 (NOT.) DATA $f:[a,b]\to\mathbb{R}$ DIREMO CHE f È INT. IN SENSO IMP. SU $[a,b]$ SE $\exists$??? $\{x_0,x_1,\dots,x_n\}\subset[a,b]$ CON $a\le x_0<\dots<x_n\le b$ t.c. $\int_{x_i}^{x_{i+1}}f(x)dx$ CONV. $\forall i=0,\dots,n-1$

ST. CONV. NEL CASO f SEGNO COSTANTE

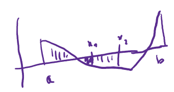
OSS. DATA $f:[a,b)\to\mathbb{R}$ t.c. $f\in\mathcal{R}([a,c])$ $\forall c\in[a,b)$

SE $f(x)\ge0$ $\forall x\in[a,b)$ ALLORA

$F(x)=\int_a^x f(t)dt$ È CRESCENTE QUINDI $\lim_{x\to b^-}F(x)$ ESISTE SEMPRE

PERCHÉ SE $x_2>x_1$ ALLORA

$$F(x_2)-F(x_1)=\int_a^{x_2}f(t)dt-\int_a^{x_1}f(t)dt=$$
$$=\int_{x_1}^{x_2}f(t)dt\ge0$$



T. 1 DATE $f,g:[a,b)\to[0,+\infty)$ t.c. $f,g\in\mathcal{R}([a,c])$ $\forall c\in[a,b)$ CON $f(x)\le g(x)$ $\forall x\in[a,b)$ ALLORA

1) $\int_a^b g(x)dx$ CONV. $\Rightarrow$ $\boxed{\int_a^b f(x)dx \text{ CONV.}}$?
2) $\int_a^b f(x)dx$ DIV. $\Rightarrow$ $\int g(x)dx$ DIV.

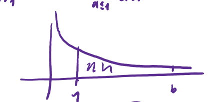
DIM.

$\lim_{c\to b^-}\int_a^c f(x)dx$? (ESISTE FINITO) SÌ ?

$$\left.\begin{array}{l}\int_a^c f(x)dx \le \boxed{\int_a^c g(x)dx}\\ \downarrow ?? \qquad\qquad \downarrow\\ \qquad\qquad\quad \lambda \text{ FINITO}\end{array}\right] \Rightarrow \text{finito}$$

ES. NOTEVOLI

1) $\int_1^{+\infty}\frac{1}{x^\alpha}dx = \begin{cases}\alpha>1 & \text{CONV.}\\ \alpha\le1 & \text{DIV.}\end{cases}$



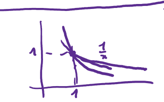
$\left(\frac{1}{1-\alpha}x^{1-\alpha}\right)' = \frac{1}{1-\alpha}\cdot$

$(x^{1-\alpha})' = x^{-\alpha}\cdot(1-\alpha)$

$\alpha\ne1$

$$\lim_{b\to+\infty}\int_1^b\frac{1}{x^\alpha}dx = \lim_{b\to+\infty}\left[\frac{1}{1-\alpha}x^{1-\alpha}\right]_1^b =$$
$$= \lim_{b\to+\infty}\frac{1}{1-\alpha}\left(b^{1-\alpha}-1\right) = \begin{cases}\alpha<1 & +\infty\\ \alpha>1 & \frac{1}{\alpha-1}\end{cases}$$

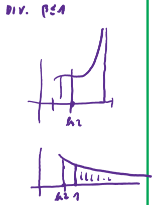
$$\underbrace{\lim_{b\to+\infty}\int_1^b\frac{1}{x}dx = \lim_{b\to+\infty}\left[\ln x\right]_1^b = \lim_{b\to+\infty}\ln(b) = +\infty}_{\frac{1}{x^\alpha}}$$



$$\int_1^{+\infty}\frac{1}{x^\alpha}dx \qquad \int_1^b\frac{1}{x}dx \overset{\forall\alpha>1}{\ge} \int_1^b\frac{1}{x^\alpha}dx = \frac{1}{\alpha-1}$$

2) $\int_2^{+\infty}\frac{1}{x^\alpha(\ln x)^\beta}dx = \begin{cases}\alpha>1 & \text{CONV. } \forall\beta\\ \alpha=1 & \begin{cases}\text{CONV. } \beta>1\\ \text{DIV. } \beta\le1\end{cases}\\ \alpha<1 & \text{DIV. } \forall\beta\end{cases}$

$\boxed{\alpha=1}$



$$\int_2^{+\infty}\frac{1}{(\ln x)^\beta}\cdot\frac{1}{x}dx =$$
$$= \lim_{b\to+\infty}\int_2^b\frac{1}{(\ln x)^\beta}\cdot(\ln x)'dx =$$
$$= \lim_{b\to+\infty}\int_{\ln2}^{\ln b}\frac{1}{y^\beta}dy =$$
$$= \lim_{b\to+\infty}\left(\int_{\ln2}^1\frac{1}{y^\beta}dy+\int_1^{\ln(b)}\frac{1}{y^\beta}dy\right) =$$
$$= \int_{\ln2}^1\frac{1}{y^\beta}dy + \boxed{\lim_{\gamma\to+\infty}\int_1^\gamma\frac{1}{y^\beta}dy}$$

$$\int_1^{+\infty}\frac{1}{y^\beta}dy = \begin{cases}\text{CONV. SE } \beta>1\\ \text{DIV. SE } \beta\le1\end{cases}$$

$\alpha<1$ $\forall\beta$

$$\int_2^{+\infty}\frac{1}{x^\alpha(\ln x)^\beta}dx \quad \text{DIV.} \qquad \delta>0$$

$\alpha=1-2\delta$

$$\frac{1}{x^\alpha(\ln x)^\beta} = \frac{1}{x^{1-2\delta}\cdot(\ln x)^\beta} = \frac{x^\delta}{x^{1-\delta}\cdot(\ln x)^\beta} = \frac{1}{x^{1-\delta}}\cdot\frac{x^\delta}{(\ln x)^\beta} \underset{x>x_0}{>} \boxed{\frac{1}{x^{1-\delta}}\cdot1}$$

$$\int_{x_0}^{+\infty} \qquad \int_{x_0}^{+\infty}$$

$$\frac{1}{x^{1+2\delta}(\ln x)^\beta} = \frac{1}{x^{1+\delta}}\cdot\frac{1}{x^\delta(\ln x)^\beta} < \frac{1}{x^{1+\delta}}$$