Analisi Matematica 2 - A.A. 21/22 - Callegari

Lezione 10

25 Marzo 2022

COME FARE

PRIMA DI INIZIARE

- RICORDARE CHE SE $f(x)\geq 0$ ALLORA $F(x)=\int_a^x f(t)\,dt$ È CRESCENTE QUINDI $\int_a^b f(t)\,dt$ NON PUÒ ESSERE INDETERMINATO.
- CONTROLLARE SE VI SONO PIÙ PUNTI IN CUI È IMPROPRIO E, SE COSÌ FOSSE, DIVIDERE L'INTERVALLO DI INTEGRAZIONE IN PIÙ PARTI, IN CIASCUNA DELLE QUALI SIA IMPROPRIO IN UN SOLO PUNTO.

USARE **T1** E **T2** PER CONFRONTARE L'INTEGRALE DA STUDIARE CON INTEGRALI GIÀ NOTI COME **I1**, **I2**, **I3** E **I4**.

T1 (CR. CONFRONTO) DATE $f,g:[a,b)\to\mathbb{R}$ T.C. $f,g\in\mathcal{R}([a,c])$ $\forall c\in[a,b)$.
SE $\forall x\in[a,b)$ SI HA $0\leq g(x)\leq f(x)$ ALLORA
1) $\int_a^b f(x)\,dx$ CONVERGE $\Rightarrow \int_a^b g(x)\,dx$ CONVERGE
2) $\int_a^b g(x)\,dx$ DIVERGE $\Rightarrow \int_a^b f(x)\,dx$ DIVERGE.

T2 (CR. CONF. ASINT.) DATE $f,g:[a,b)\to(0,+\infty)$ T.C. $f,g\in\mathcal{R}([a,c])$ $\forall c\in[a,b)$, ALLORA:
$\left(f(x)\simeq g(x) \text{ PER } x\to b^-\right) \Rightarrow \left(\int_a^b f(x)\,dx \text{ E } \int_a^b g(x)\,dx \text{ HANNO LO STESSO CARATTERE}\right)$

I1 $\int_1^{+\infty}\frac{1}{x^\alpha}\,dx$ = CONVERGE SE $\alpha>1$; DIVERGE SE $\alpha\leq 1$

I2 $\int_0^{1}\frac{1}{x^\alpha}\,dx$ = CONVERGE SE $\alpha<1$; DIVERGE SE $\alpha\geq 1$

I3 $\int_2^{+\infty}\frac{1}{x^\alpha(\ln x)^\beta}\,dx$ = $\alpha>1$: CONVERGE $\forall\beta\in\mathbb{R}$; $\alpha=1$: $\beta>1$ CONVERGE, $\beta\leq 1$ DIVERGE; $\alpha<1$: DIVERGE $\forall\beta\in\mathbb{R}$

I4 $\int_0^{\frac12}\frac{1}{x^\alpha|\ln x|^\beta}\,dx$ = $\alpha<1$: CONVERGE $\forall\beta\in\mathbb{R}$; $\alpha=1$: $\beta>1$ CONVERGE, $\beta\leq 1$ DIVERGE; $\alpha>1$: DIVERGE $\forall\beta\in\mathbb{R}$

PER SISTEMARE I DETTAGLI SONO UTILI LE SEGUENTI OSSERVAZIONI:

OS.1 DATA $f:[a,b)\to\mathbb{R}$ T.C. $f\in\mathcal{R}([a,c])$ $\forall c\in\mathbb{R}$ E SIA $x_0\in(a,b)$ ALLORA $\int_a^b f(x)\,dx$ E $\int_{x_0}^b f(x)\,dx$ HANNO STESSO CARATTERE.

OS.2 $W=\left\{f:[a,b)\to\mathbb{R} \mid f\in\mathcal{R}([a,c])\ \forall c\in[a,b) \text{ E } \int_a^b f(x)\,dx \text{ CONVERGE}\right\}$ È UNO SP. VETTORIALE

OS.3 $g:[a,b)\to\mathbb{R}$ T.C. $g\in\mathcal{R}([a,c])$ $\forall c\in[a,b)$ E SIA W COME IN **OS.2**
ALLORA $\int_a^b g(x)\,dx$ E $\int_a^b \left(g(x)+f(x)\right)$ HANNO LO STESSO CARATTERE. $f\in W$

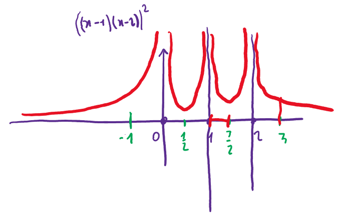
OS.4 SIA $f:[a,b)\to[0,+\infty)$ T.C. $f\in\mathcal{R}([a,c])$ $\forall c\in[a,b]$
E SIA (a_n) A VALORI IN (a,b) TALE CHE $a_n\to b^-$.
ALLORA: $\lim_{n\to+\infty}\int_a^{a_n} f(x)\,dx=\int_a^b f(x)\,dx$

SE PERÒ NULLA DI TUTTO QUESTO FUNZIONA BISOGNA INVENTARSI QUALCOSA.

ESEMPI DI STUDIO CONVERGENZA CON INTEGRANDA POSITIVA.

1. $\int_0^{+\infty}\frac{x\ln\left(1+\frac{1}{x^2}\right)}{\ln^3(1+\sqrt{x})}\,dx$
2. $\int_0^{+\infty}\frac{\left(1+\frac1x\right)^x}{\ln(1+x^x)}\,dx$
3. $\int_0^{+\infty}\left|\frac{\sin x}{x}\right|dx$
4. $\int_0^{+\infty}\left|\lambda+\frac{\sin x}{x}\right|^x dx$ $(\lambda>0)$
5. $\int_0^{+\infty}\left|\frac{\lambda+\sin x}{2}\right|^x dx$ $(\lambda>0)$
6. $\int_0^{+\infty}\chi_A(x)\,dx$ CON $A=\bigcup_{n=1}^{+\infty}\left[n-\frac{1}{n^2},n\right]$
7. $\int_{-\infty}^{+\infty}\frac{e^{\sin x}}{\sqrt[3]{(x^2-3x+2)^2\cdot\ln(1+x^2)}}\,dx$

$\underbrace{(x^2-3x+2)}_{}$ $\left((x-1)(x-2)\right)^2$



$\int_{-\infty}^{-1}+\int_{-1}^{0}+\int_0^{\frac12}+\int_{\frac12}^{1}+\int_1^{\frac32}+\int_{\frac32}^{2}+\int_2^3+\int_3^{+\infty}$

$x\to1^+$

$\int_1^{\frac32}\frac{e^{\sin x}}{\sqrt[3]{(x-1)^2(x-2)^2\cdot\ln(1+x^2)}}\,dx$; $e^{\sin x}\to e^{\sin 1}$; $(x-2)^2\to 1$; $\ln(1+x^2)\to\ln 2$

$\simeq\frac{e^{\sin 1}}{\sqrt[3]{\ln 2}}\cdot\frac{1}{|x-1|^{\frac23}}$

$\int_1^{\frac32}\frac{1}{|x-1|^{\frac23}}\,dx=\ldots$

$\ldots=\int_0^{\frac12}\frac{1}{|y|^{\frac23}}\,dy=\int_0^{\frac12}\frac{1}{y^{\frac23}}\,dy$

$\boxed{\int_0^1\frac{1}{y^{\frac23}}\,dy}$ CONV.

1

$\int_0^{+\infty}\underbrace{\frac{x\ln\left(1+\frac{1}{x^2}\right)}{\ln^3(1+\sqrt{x})}}_{f(x)}dx$; $\ln^3(1+\sqrt{x})\simeq x^{\frac32}$ per $x\to0^+$

$x\to0^+$

$f(x)\approx\frac{x\cdot\ln\left(\frac{1}{x^2}\cdot(x^2+1)\right)}{x^{\frac32}}=$

$=\frac{1}{\sqrt{x}}\cdot\left(-2\ln x+\ln(1+x^2)\right)\approx$

$-2\ln\frac{1}{\sqrt{x}}$ $\approx-\frac{2}{\sqrt{x}}\cdot\ln x=\frac{2}{\sqrt{x}}\cdot|\ln x|=$ $x<1$

$=2\cdot\frac{1}{x^{\frac12}\cdot|\ln x|^{-1}}$ $\frac{1}{x^\alpha|\ln x|^\beta}$

$\underbrace{\int_0^{\frac12}f(x)\,dx}_{1}+\underbrace{\int_{\frac12}^{+\infty}f(x)\,dx}_{2}$ CONV.

1 $\int_0^{\frac12}\frac{x\ln\left(1+\frac{1}{x^2}\right)}{\ln^3(1+\sqrt{x})}\,dx$ $\approx2\cdot\frac{1}{x^{\frac12}\cdot|\ln x|^{-1}}$

CONV. PER **T.2** $2\int_0^{\frac12}\frac{1}{x^{\frac12}\cdot|\ln x|^{-1}}\,dx$ CONV.

$\int_{\frac12}^{+\infty}\frac{x\ln\left(1+\frac{1}{x^2}\right)}{\ln^3(1+\sqrt{x})}\,dx$

$\int_2^{+\infty}\frac{x\ln\left(1+\frac{1}{x^2}\right)}{\ln^3(1+\sqrt{x})}\,dx$ ← CONV. **T.2** ; $\approx\frac1x$ $(x\to+\infty)$; $\approx\frac18\ln^3 x$

$\ln^3(1+\sqrt{x})\approx(\ln\sqrt{x})^3=\frac18\ln^3 x$

$\ln(1+\sqrt{x})=\ln\left(\sqrt{x}\cdot\left(\frac{1}{\sqrt{x}}+1\right)\right)=$
$=\ln\sqrt{x}+\ln\left(1+\frac{1}{\sqrt{x}}\right)\approx\ln\sqrt{x}=\frac12\ln x$

$\approx\frac{\frac1x}{\frac18\ln^3 x}=\boxed{\frac{8}{x\ln^3 x}}$ $8\int_2^{+\infty}\frac{1}{x\cdot(\ln x)^3}\,dx$ CONV.

$\int_{\frac12}^{+\infty}\frac{1}{x\cdot(\ln x)^3}\,dx$; 2 ; 1,1 ; 1

2

$\int_0^{+\infty}\frac{\left(1+\frac1x\right)^x}{\ln(1+x^x)}\,dx$ $\underset{x\to+\infty}{\approx}\frac{e}{\ln(x^x)}=\frac{e}{x\ln x}$; $\int_2^{+\infty}\frac{1}{x^\alpha(\ln x)^\beta}$

$\lim_{x\to0^+}\left(1+\frac1x\right)^x=\lim_{x\to0^+}e^{x\ln\left(1+\frac1x\right)}=e^0=1$

$\lim_{x\to0^+}x\ln\left(1+\frac1x\right)=\lim_{x\to0^+}x\ln\left(\frac1x\right)=$

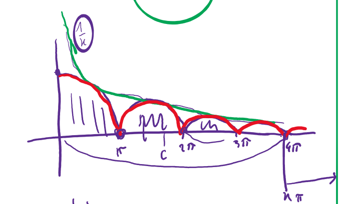
$=\lim_{x\to0^+}\frac{1}{\frac1x}\cdot\ln\left(\frac1x\right)=$

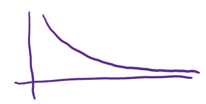
$y=\frac1x$ $=\lim_{y\to+\infty}\frac{\ln y}{y}=0$

$\int_2^{+\infty}\frac{\left(1+\frac1x\right)^x}{\ln(1+x^x)}\,dx$ $\longleftrightarrow$ ST. CARATT. **T.2** $\int_2^{+\infty}\frac{1}{x\ln x}\,dx$ IV.

3

$\int_0^{+\infty}\left|\frac{\sin x}{x}\right|dx$; $|\sin x|\cdot\frac1x$





$\lim_{c\to+\infty}\int_0^c\left|\frac{\sin x}{x}\right|dx=+\infty$?

$\lim_{n\to+\infty}\int_0^{n\pi}\left|\frac{\sin x}{x}\right|dx=$

$=\lim_{n\to+\infty}\left(\int_0^{\pi}\left|\frac{\sin x}{x}\right|dx+\int_{\pi}^{2\pi}\ldots dx+\ldots+\int_{(n-1)\pi}^{n\pi}\left|\frac{\sin x}{x}\right|dx\right)=$
$\frac{2}{\pi}\cdot\frac11$ $\quad\frac{2}{\pi}\cdot\frac12\ \ldots\ \ldots+\frac{2}{\pi}\cdot\frac1n$

$=\lim_{n\to+\infty}\sum_{k=1}^{n}\underbrace{\left(\int_{(k-1)\pi}^{k\pi}\left|\frac{\sin x}{x}\right|dx\right)}_{a_k}\geq\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{2}{\pi}\cdot\frac1k=$ $=\frac{2}{\pi}\lim_{n\to+\infty}\sum_{k=1}^{n}\frac1k$

$=$ $\frac{|\sin x|}{x}\geq\frac{|\sin x|}{k\pi}$

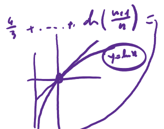
$a_k=\int_{(k-1)\pi}^{k\pi}\left|\frac{\sin x}{x}\right|dx\geq\int_{(k-1)\pi}^{k\pi}\frac{|\sin x|}{k\pi}=\frac{1}{k\pi}\int_{(k-1)\pi}^{k\pi}|\sin x|\,dx=$

$=\frac{1}{k\pi}\int_0^{\pi}|\sin x|\,dx=\frac{2}{k\pi}=\frac{2}{\pi}\cdot\frac1k$

$=\frac{2}{\pi}\lim_{n\to+\infty}\left(1+\frac12+\frac13+\ldots+\frac1n\right)=+\infty$

$\geq\ln(1+1)+\ln\left(1+\frac12\right)+\ln\left(1+\frac13\right)+\ldots+\ln\left(1+\frac1n\right)=\ln(n+1)\to+\infty$

$\ln2+\ln\frac32+\ln\frac43+\ldots+\ln\left(\frac{n+1}{n}\right)=$



$x>\ln(1+x)$

$\frac1n>\ln\left(1+\frac1n\right)$ $\forall n$

$\ln\left(2\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdots\frac{n+1}{n}\right)=\ln(n+1)$

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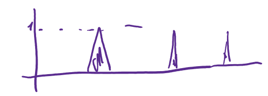
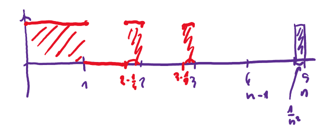


$\int_0^{+\infty}\chi_A(x)\,dx$

$\bigcup_{n=1}^{+\infty}\left[n-\frac{1}{n^2},n\right]$



$\lim_{n\to+\infty}\int_0^n f(x)\,dx=\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{k^2}=$ [illegible] ?

$\frac11+\frac14+\frac19+\frac1{16}+\ldots+\frac{1}{n^2}=$

$\frac11+\frac{1}{2\cdot2}+\frac{1}{3\cdot3}+\frac{1}{4\cdot4}+\ldots+\frac{1}{n\cdot n}\leq$

$\leq1+\frac{1}{2\cdot1}+\frac{1}{3\cdot2}+\frac{1}{4\cdot3}+\ldots+\frac{1}{n(n-1)}=$

$=1+\frac{2-1}{2\cdot1}+\frac{3-2}{3\cdot2}+\frac{4-3}{4\cdot3}+\ldots+\frac{n-(n-1)}{n(n-1)}=$

$=1+1-\cancel{\frac12}+\cancel{\frac12}-\cancel{\frac13}+\cancel{\frac13}-\cancel{\frac14}+\ldots+\cancel{\frac{1}{n-1}}-\frac1n=$

$=1+1-\frac1n\to2$