Analisi Matematica 2 - A.A. 21/22 - Callegari

Lezione 11

28 Marzo 2022

1) CRIT. ASSOLUTA INTEGRABILITÀ
2) ESEMPIO CATTIVO $\int_1^{+\infty} \frac{\sin x}{x}\,dx$
3) ESTENSIONI ES. CATTIVO $\int_1^{+\infty} \sin x^2 \quad \int_1^{+\infty} x\cdot\sin x^3\,dx$
4) CRIT. INT. OSCILLANTI
5) ESEMPI: $\int_1^{+\infty} \frac{\sin(\sin x)}{\ln(1+x^2)}\,dx \quad \int_1^{+\infty} \frac{\sin(x^2)}{2+\ln x}\,dx$
6) ESEMPIO ANCORA PIÙ CATTIVO: $\int_1^{+\infty} \frac{\sin x}{\sqrt{x}+\sin x}\,dx$
7) CRITERIO DI CAUCHY
8) RESTO

INTEGRALI IMPROPRI

(INTEGRANDA DI SEGNO VARIABILE)

[CR. ASS. CONV.]

T.1 DATA $f:[a,b)\to\mathbb{R}$ T.C. $f\in\mathcal{R}([a,c])\ \forall c\in[a,b)$

SUPPONIAMO INOLTRE CHE $\int_a^b |f(x)|\,dx$ CONVERGA

ALLORA ANCHE $\int_a^b f(x)\,dx$ CONVERGE.

DIM

PRENDO $f^+(x)$ E $f^-(x)$

$f(x)=f^+(x)-f^-(x)$

$|f(x)|=f^+(x)+f^-(x)$

$0\le f^+(x)\le |f(x)|$

$0\le f^-(x)\le |f(x)|$

T. CONF.

$\int_a^b f^+(x)\,dx$ CONV.

$\int_a^b f^-(x)\,dx$ CONV.

$\Downarrow$

$\int_a^b f(x)\,dx=\int_a^b \left(f^+(x)-f^-(x)\right)dx$ CONV.

OSS. $\int |f|$ CONV $\Rightarrow \int f$ CONV.

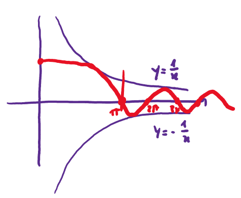
MA NON VICEVERSA:

$\int |f|$ DIV. $\not\Rightarrow \int f$ DIV

ES.1 $f(x)=\frac{\sin x}{x}$

$\int_0^{+\infty} \left|\frac{\sin x}{x}\right| dx$ DIV. (VISTO VOLTA SCORSA)

MA $\int_0^{+\infty} \frac{\sin x}{x}\,dx$ CONV.



BASTA STUDIARE

$\int_\pi^{+\infty} \frac{\sin x}{x}\,dx =$

$=\lim_{c\to+\infty}\int_\pi^c \frac{\sin x}{x}\,dx =$

$=\lim_{c\to+\infty}\int_\pi^c \frac{1}{x}\cdot(-\cos x)'\,dx =$

$=\lim_{c\to+\infty}\left(\left[-\frac{\cos x}{x}\right]_\pi^c-\int_\pi^c \frac{1}{x^2}\cos x\,dx\right)$

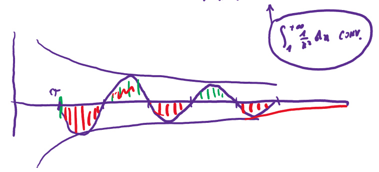
$=\lim_{c\to+\infty}\left(-\frac{\cos c}{c}+\frac{\cos\pi}{\pi}-\int_\pi^c \frac{\cos x}{x^2}\,dx\right)=$

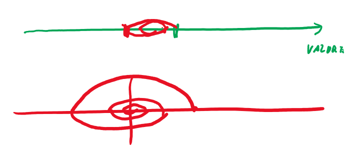
$= -\frac{1}{\pi}-\underbrace{\int_\pi^{+\infty}\frac{\cos x}{x^2}\,dx}_{\text{CONV. PERCHÉ}}$

$\int_\pi^{+\infty}\left|\frac{\cos x}{x^2}\right|dx$ CONV. PER CONFRONTO

IN QUANTO $0\le\left|\frac{\cos x}{x^2}\right|\le\frac{1}{x^2}$

$\int_1^{+\infty}\frac{1}{x^2}\,dx$ CONV.

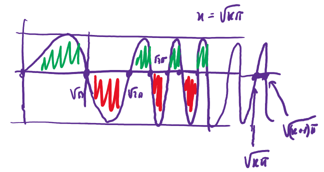




ES.2 $\int_0^{+\infty}\sin(x^2)\,dx$

$x^2=k\pi$

$x=\sqrt{k\pi}$



$\frac{\sqrt{(k+1)\pi}-\sqrt{k\pi}}{1}=\frac{\pi}{\sqrt{(k+1)\pi}+\sqrt{k\pi}}\to 0$

$\int_{\sqrt{\pi}}^{+\infty}\sin(x^2)\,dx=\lim_{c\to+\infty}\int_{\sqrt{\pi}}^c \sin(x^2)\,dx \overset{x=\sqrt{t}}{=}$

$=\lim_{c\to+\infty}\int_\pi^{c^2}\sin\left((\sqrt{t})^2\right)\cdot(\sqrt{t})'\,dt=$

$=\lim_{c\to+\infty}\int_\pi^{c^2}\frac{\sin t}{2\sqrt{t}}\,dt=$

$=\lim_{c\to+\infty}\int_\pi^{c^2}\frac{1}{2\sqrt{t}}\cdot(-\cos t)'\,dt=$

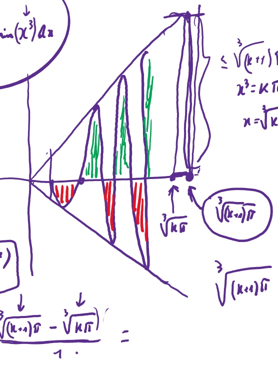
$=\lim_{c\to+\infty}\left(\left[\frac{-\cos t}{2\sqrt{t}}\right]_\pi^{c^2}-\int_\pi^{c^2}\frac{1}{4}\frac{\cos t}{t\sqrt{t}}\,dt\right)$

$=-\frac{1}{2\sqrt{\pi}}-\frac{1}{4}\underbrace{\int_\pi^{+\infty}\frac{\cos t}{t^{\frac{3}{2}}}\,dt}_{\text{CONV.}}$

$\Uparrow$

$0\le\left|\frac{\cos t}{t^{\frac{3}{2}}}\right|\le\frac{1}{t^{\frac{3}{2}}}$ E $\int_\pi^{+\infty}\frac{1}{t^{\frac{3}{2}}}\,dt$ CONV $\Rightarrow \int_\pi^{+\infty}\left|\frac{\cos t}{t^{\frac{3}{2}}}\right|dt$ CONV.

ES.3 $\int_{\sqrt[3]{\pi}}^{+\infty} x\sin(x^3)\,dx$



$\sqrt[3]{(k+1)\pi}-\sqrt[3]{k\pi}$

$(a-b)(a^2+ab+b^2)$

a^3-b^3

$\frac{\left(\sqrt[3]{(k+1)\pi}-\sqrt[3]{k\pi}\right)}{1}=$

$=\frac{\pi}{\left((k+1)\pi\right)^{\frac{2}{3}}+\left((k\pi)(k+1)\pi\right)^{\frac{1}{3}}+(k\pi)^{\frac{2}{3}}}=$

$=\frac{\pi}{k^{2/3}}$

$\int_{\sqrt[3]{\pi}}^{+\infty}x\cdot\sin(x^3)\,dx=\lim_{c\to+\infty}\int_{\sqrt[3]{\pi}}^c x\sin(x^3)\,dx \overset{x=\sqrt[3]{t}}{=}$

$=\lim_{c\to+\infty}\int_\pi^{c^3}\sqrt[3]{t}\cdot\sin t\cdot\frac{1}{3\sqrt[3]{t^2}}\,dt=$

$=\lim_{c\to+\infty}\int_\pi^{c^3}\frac{\sin t}{3\sqrt[3]{t}}\,dt=\dots$ COME PRIMA.

$\int\frac{g(t)}{f(t)}\,dt$

$\int_a^{+\infty}g(x)f(x)\,dx$

T.2 DATE $f,g:[a,+\infty)\to\mathbb{R}$ T.C.

- $g(x)\in C([a,+\infty))$ t.c. $\exists\, G:[a,+\infty)\to\mathbb{R}$ LIMITATA E TALE CHE $G'(x)=g(x)$
- $f\in C^1([a,+\infty))$ t.c. $f(x)\to 0$ DECRESC. PER $x\to+\infty$

ALLORA $\int_a^{+\infty}f(x)g(x)\,dx$ CONVERGE.

DIM

DEVO MOSTRARE CHE ES. FINITO

$\lim_{c\to+\infty}\int_a^c f(x)\cdot g(x)\,dx=$

$=\lim_{c\to+\infty}\int_a^c f(x)\cdot\left(G(x)\right)'\,dx=$

$=\lim_{c\to+\infty}\left(\left[G(x)\cdot f(x)\right]_a^c-\int_a^c f'(x)G(x)\,dx\right)=$

$=\lim_{c\to+\infty}\left(\underbrace{G(c)\cdot f(c)}_{0}-G(a)f(a)-\int_a^c f'(x)G(x)\,dx\right)=$

$=G(a)f(a)-\boxed{\int_a^{+\infty}f'(x)G(x)\,dx}$ CONV?

CR. ASS. CONV.

VEDO SE CONV. $\int_a^{+\infty}\left|f'(x)\cdot G(x)\right|dx$ CONV. PER C. CON

SIA $M>0$ t.c. $\forall x\in[a,+\infty)$ $|G(x)|\le M$ (TALE M ESISTE PERCHÉ G(x) È LIMITATA)

$0\le\left|f'(x)\cdot G(x)\right|\le\left|f'(x)\right|\cdot M=$

$=-f'(x)\cdot M$

BASTA MOSTRARE CHE CONV.

$\boxed{\int_a^{+\infty}-f'(x)\cdot M\,dx}=$

$=\lim_{c\to+\infty}-M\int_a^c f'(x)\,dx=$

$=\lim_{c\to+\infty}-M\cdot\left[f(x)\right]_a^c=$

$=\lim_{c\to+\infty}\left(-M f(c)+M f(a)\right)=M\cdot f(a)$

T.3 (CR. DI CAUCHY)

DATA $f:[a,b)\to\mathbb{R}$ t.c. $f\in\mathcal{R}([a,c])\ \forall c\in[a,b)$.

ALLORA SONO EQUIV. LE 2 SEGUENTI AFFERMAZIONI:

1) $\int_a^b f(x)\,dx$ CONVERGE

2) $\forall\varepsilon>0\ \exists\, M\in(a,b)$ T.C. $\forall x_1,x_2\in(M,b)$ SI HA $\left|\int_{x_1}^{x_2}f(x)\,dx\right|<\varepsilon$

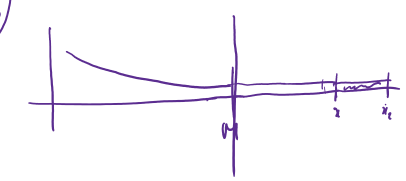
DIM

$F(x)=\int_a^x f(t)\,dt$

(1) $\Leftrightarrow \lim_{x\to+\infty}F(x)$ ES. FINITO

$\int_{x_1}^{x_2}f(x)\,dx=\int_a^{x_2}-\int_a^{x_1}=F(x_2)-F(x_1)$

(2) $\forall\varepsilon>0\ \exists\, M\in(a,b)$ t.c. $\forall x_1,x_2>M$ $\left|F(x_1)-F(x_2)\right|<\varepsilon$



$\int_0^{+\infty}\sin x$