Analisi Matematica 2 - A.A. 21/22 - Callegari

Lezione 13

1 Aprile 2022

SERIE NUMERICHE

1) ESEMPI $\sum \frac{1}{2^n}$, $\sum \frac{1}{n}$, $\sum \frac{1}{n(n+1)}$, $\sum (-1)^n$

2) DEFINIZIONE

3) ESEMPI : $\sum q^n$, $0{,}2\overline{16} = ?$

4) OSS. SU PROPRIETÀ COMMUTATIVA ASSOCIATIVA E DISTRIBUTIVA

5) PROPRIETÀ BASE =
   1) COND. NECESSARIA
   2) COMB. LINEARI DI SERIE CONV.
   3) TRASFORMAZIONI CHE NON CAMBIANO IL CARATTERE
   4) BONTÀ DI SERIE A TERMINI POSITIVI

6) CRIT. DI CAUCHY

ES.1

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots \quad = 2$$

$$1 \qquad 1 = 2-1$$
$$1+\frac{1}{2} \qquad \frac{3}{2} = 2-\frac{1}{2}$$
$$1+\frac{1}{2}+\frac{1}{4} \qquad \frac{7}{4} = 2-\frac{1}{4}$$
$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8} \qquad \frac{15}{8} = 2-\frac{1}{8}$$
$$\vdots$$
$$\boxed{1+\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^n}} = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = \boxed{2-\left(\frac{1}{2}\right)^n}$$

$$1+x+x^2+\dots+x^n = \frac{1-x^{n+1}}{1-x}$$
$$(1-x)(1+x+\dots+x^n) =$$
$$= 1 + \cancel{x} + \cancel{x^2} + \dots + \cancel{x^n} - \cancel{x} - \cancel{x^2} - \cancel{x^3} - \dots - x^{n+1} =$$
$$= 1 - x^{n+1}$$

$$S_n = a_0 + a_1 + a_2 + \dots + a_n$$
$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \left(2-\left(\frac{1}{2}\right)^n\right) = 2$$

ES.2

$$a_n = (-1)^n$$
$$\to \quad 1 + (-1) + 1 + (-1) + 1 + (-1) + \dots$$
$$1 + 1 + (-1) + 1 + 1 + (-1)$$

DEF. DATA (a_n) A VALORI IN $\mathbb{R}$, DIREMO CHE

1) " $\sum_{n=n_0}^{+\infty} a_n = \ell \in \mathbb{R}$ " SE $\lim_{n\to+\infty} S_n = \ell$ DOVE $S_n = a_{n_0} + \dots + a_n$

2) " $\sum_{n=n_0}^{+\infty} a_n = +\infty$ " SE $\lim_{n\to+\infty} S_n = +\infty$ . . . .

3) " $\sum_{n=n_0}^{+\infty} a_n$ È INDETERMINATA" SE $\lim_{n\to+\infty} S_n$ NON ESISTE.

ES.2 (CONTINUA)

$$a_n = (-1)^n$$
$\sum_{n=0}^{+\infty} a_n$ È INDET.

$$S_n = a_0 + a_1 + \dots + a_n = \begin{cases} 0 & n \text{ dispari} \\ 1 & n \text{ pari} \end{cases}$$

QUINDI $\lim_{n\to+\infty} S_n$ NON ESISTE

ES.3

$$a_n = \frac{1}{n}$$

$$\boxed{\sum_{n=1}^{+\infty} \frac{1}{n} \text{ DIVERGE}} \qquad \boxed{\int_1^{+\infty} \frac{1}{x}\,dx \text{ diverge}}$$

$\boxed{a_n \geq 0}$

$$S_n = a_1 + a_2 + \dots + a_n$$

(S_n)
$$S_{n+1} = \underbrace{a_1 + a_2 + \dots + a_n}_{S_n} + a_{n+1}$$
$$S_{n+1} = S_n + a_{n+1}$$

(S_n) $\quad S_n \to +\infty$

(S_{n_k}) $\quad S_{2^k} \to +\infty$

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$$

$$S_{2^k} = 1 + \underbrace{\frac{1}{2}}_{k=1} + \underbrace{\left(\frac{1}{3}+\frac{1}{4}\right)}_{k=2} + \dots + \overbrace{\left(\frac{1}{2^{k-1}+1} + \dots + \frac{1}{2^k}\right)}^{\geq 2^{k-1}\cdot\frac{1}{2^k} = \frac{1}{2}} \geq 1 + \frac{k}{2}$$

$$S_{16} = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3}+\frac{1}{4}\right)}_{\geq\frac{1}{2}} + \underbrace{\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}_{\geq\frac{1}{2}} + \underbrace{\left(\frac{1}{9}+\dots+\frac{1}{16}\right)}_{\frac{1}{2}} + \underbrace{\left(\frac{1}{17}+\dots+\frac{1}{32}\right)}_{\frac{1}{2}}$$

S_{32}

$$S_{2^k} \geq 1 + \frac{k}{2}$$
$$\downarrow \qquad \downarrow$$
$$+\infty \qquad +\infty$$

ES.4 $\sum_{n=1}^{+\infty} \underbrace{\frac{1}{n(n+1)}}_{a_n}$ (MENGOLI)

$$S_n = a_1 + a_2 + \dots + a_n =$$

$$\lim_{n\to+\infty} S_n = \dots$$

$$S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{n\cdot(n+1)} =$$
$$= \frac{2-1}{1\cdot 2} + \frac{3-2}{2\cdot 3} + \frac{4-3}{3\cdot 4} + \dots + \frac{n+1-n}{n\cdot(n+1)} =$$
$$= \frac{\cancel{2}}{1\cdot\cancel{2}} - \frac{1}{1\cdot 2} + \frac{\cancel{3}}{2\cdot\cancel{3}} - \frac{\cancel{2}}{\cancel{2}\cdot 3} + \frac{\cancel{4}}{3\cdot\cancel{4}} - \frac{\cancel{3}}{\cancel{3}\cdot 4} + \dots + \frac{\cancel{n+1}}{n\cdot\cancel{(n+1)}} - \frac{\cancel{n}}{\cancel{n}\cdot(n+1)} =$$
$$= \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \dots + \cancel{\frac{1}{n}} - \frac{1}{n+1} =$$
$$= 1 - \frac{1}{n+1}$$

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \left(1 - \frac{1}{n+1}\right) = 1$$

ES.5

$$\sum_{n=0}^{+\infty} p^n = \begin{cases} \frac{1}{1-p} & |p|<1 \\ +\infty & p \geq 1 \\ \text{IND.} & p \leq -1 \end{cases}$$

$\boxed{p=1}$ $\quad S_n = \overbrace{1+1+1+\dots+1}^{n+1} = n+1$

$\lim_{n\to+\infty} S_n =$

$p \neq 1$
$$S_n = 1 + p + p^2 + \dots + p^n = \frac{1-p^{n+1}}{1-p} = \frac{1}{1-p} - \frac{1}{1-p}\cdot p^{n+1}$$

$$\lim_{n\to+\infty}\left(\frac{1}{1-p} - \frac{1}{1-p}\cdot p^{n+1}\right) = \begin{cases} \frac{1}{1-p} & |p|<1 \\ +\infty & p>1 \\ \text{NON ESISTE} & p \leq -1 \end{cases}$$

NON ESISTE

ES.6

$$5{,}3\,16\,16\,16\,16$$
$$5{,}3\overline{16} = \qquad \sum p^n \quad \left(p = \frac{1}{100}\right)$$

$$= 5 + \frac{3}{10} + \frac{16}{10^3} + \frac{16}{10^5} + \frac{16}{10^7} + \frac{16}{10^9} + \dots + \frac{16}{10^{2k+1}} + \dots$$

$$= 5 + \frac{3}{10} + \frac{16}{10^3}\cdot\left(1 + \frac{1}{100} + \frac{1}{100^2} + \dots + \frac{1}{100^n} + \dots\right) =$$

$$= 5 + \frac{3}{10} + \frac{16}{1000}\cdot\frac{1}{1-\frac{1}{100}} =$$

$$= 5 + \frac{3}{10} + \frac{16}{\cancel{1000}_{10}}\cdot\frac{\cancel{100}}{99} = 5 + \frac{3}{10} + \frac{16}{990}$$

T.1 (COND. NEC. DI CONV.)

DATA (a_n), SE $\sum a_n$ CONV. $\Rightarrow a_n \to 0$

$\Leftarrow$ NON VALE $\sum \frac{1}{n}$

DIM

$$\lim_{n\to+\infty} S_n = \ell \in \mathbb{R}$$

$$a_n = S_n - S_{n-1} \to \ell - \ell = 0$$
$$\quad \downarrow \quad \downarrow$$
$$\quad \ell \quad \ell$$

T.2 DATE (a_n) E (b_n) t.c. $\sum a_n$ E $\sum b_n$ CONV.

ALLORA $\forall \alpha, \beta \in \mathbb{R}$ $\sum(\alpha a_n + \beta b_n)$ CONV.

DIM

$\to S_n = a_0 + a_1 + \dots + a_n \qquad S_n \to \ell \in \mathbb{R}$

$\to \sigma_n = b_0 + b_1 + \dots + b_n \qquad \sigma_n \to L \in \mathbb{R}$

$$\sum_{n=0}^{+\infty}(\alpha a_n + \beta b_n)$$

$$\mathcal{S}_n = (\alpha a_0 + \beta b_0) + (\alpha a_1 + \beta b_1) + \dots + (\alpha a_n + \beta b_n) = \alpha(a_0 + a_1 + \dots + a_n) + \beta(b_0 + b_1 + \dots + b_n)$$

$$\lim_{n\to+\infty} \mathcal{S}_n = \lim_{n\to+\infty}(\alpha S_n + \beta \sigma_n) = \alpha\ell + \beta L \in \mathbb{R}$$

QUINDI CONV.

T.3 DATE (a_n) E (b_n) t.c. $\sum a_n$ CONV.

ALLORA $\sum b_n$ E $\sum(b_n + a_n)$ HANNO STESSO CARATTERE

DIM

$$S_n = a_0 + a_1 + \dots + a_n$$
$$\sigma_n = b_0 + \dots + b_n$$
$$\mathcal{S}_n = (a_0 + b_0) + (a_1 + b_1) + \dots + (a_n + b_n)$$

$$\lim_{n\to+\infty} \mathcal{S}_n = \lim_{n\to+\infty}(S_n + \sigma_n) = \left(\text{FA STESSA COSA DI } \sigma_n\right)$$

T.4 DATA (a_n) E $\lambda \in \mathbb{R} - \{0\}$ ALLORA

$\sum a_n$ E $\sum \lambda a_n$ HANNO ST. CARATTERE

E, SE CONVERGONO

$$\sum \lambda a_n = \lambda \sum a_n$$

DIM

$$S_n = a_0 + a_1 + \dots + a_n$$
$$\mathcal{S}_n = \lambda a_0 + \lambda a_1 + \dots + \lambda a_n = \lambda \cdot S_n$$

T.5

$$\sum_{n=1}^{+\infty} a_n \qquad \sum_{n=n_0}^{+\infty} a_n$$

T.6

$$S_n = a_1 + a_2 + \dots + a_n$$
$$\mathcal{S}_n = a_{n_0} + a_{n_0+1} + \dots + a_n$$

$$S_n = a_1 + \dots + a_{n_0-1} + \overbrace{(a_{n_0} + \dots + a_n)}^{\mathcal{S}_n}$$

$$\boxed{S_n = \overbrace{a_1 + \dots + a_{n_0-1}}^{A} + \mathcal{S}_n}$$

$$\mathcal{S}_n = -A + S_n$$