Analisi Matematica 2 - 4 Aprile 2022 - docente: Callegari

Lezione 14

SERIE A TERMINI POSITIVI

1) CRITERI
- CONFRONTO
- CONF. ASINTOTICO
- INTEGRALE (+EX)
- RADICE
- RAPPORTO

ESEMPI

1. $\sum \frac{\cos^4 n + \sin^4 n}{n}$
2. $\sum \frac{1}{n^2}$
3. $\sum \frac{1}{(\ln n)^{\ln n}}$
4. $\sum \frac{1}{(\ln n)^{\ln(\ln n)}}$
5. $\sum \left(1 - \cos \frac{1}{n}\right)$

$\boxed{\sum_{n=1}^{+\infty} \frac{1}{n^\alpha} \begin{cases} \text{CONV. } \alpha > 1 \\ \text{DIV. } \alpha \leq 1 \end{cases}}$

$\sum_{n=1}^{+\infty} \frac{1}{n^\alpha (\ln n)^\beta}$

→ 6 7 [...]

8. $\sum \frac{1}{\ln(n!)}$
9. $\sum \frac{n! \cdot n^{n+1}}{(2n)!} \cdot A^n$

$\boxed{T.1}$ (CR. CONFRONTO)

DATE $(a_n), (b_n)$ SUCC. A TERMINI ≥ 0 T.C.

(*) $\boxed{a_n \geq b_n \text{ DEF. IN } n}$ ALLORA SI HA:

⇒ 1) $\sum a_n$ CONV. $\Rightarrow \sum b_n$ CONV

2) $\sum b_n$ DIV. $\Rightarrow \sum a_n$ DIV.

$\boxed{\text{DIM}}$ POSSIAMO SUPPORRE CHE (*) VALGA $\forall n \in \mathbb{N}$ PERCHÉ POSSIAMO CAMBIARE UN NUM. FINITO DI TERMINI SENZA CHE IL CARATTERE CAMBI.

SIANO $S_n = a_0 + a_1 + \dots + a_n$

$\sigma_n = b_0 + b_1 + \dots + b_n$

SI HA $S_n \geq \sigma_n$ $\forall n$ PERCHÉ

$$S_n = a_0 + \dots + a_n \geq b_0 + \dots + b_n = \sigma_n$$

QUINDI

$\sum a_n$ CONV $\Rightarrow \lim_{n\to+\infty} S_n = \lambda \in \mathbb{R}$

$\boxed{\forall n \in \mathbb{N} \quad \lambda \geq S_n \geq \sigma_n}$

$\Downarrow$

$\lim_{n\to+\infty} \sigma_n$ È $\leq \lambda$ QUINDI FINITO

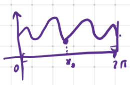
QUINDI $\sum b_n$ CONVERGE.

(2) SEGUE DA (1).

$\boxed{ES.1}$ $\sum_{n=1}^{+\infty} \frac{\sin^4 n + \cos^4 n}{n}$

$\boxed{\frac{\sin^4 n + \cos^4 n}{n}} \geq \frac{c}{n}$ $(c>0)$

$f(x) = \boxed{\cos^4 x + \sin^4 x}$



$\forall x \in [0, 2\pi]$ $f(x) \geq f(x_0)$

$\boxed{ES.2}$ $\sum \frac{1}{n^2}$ $\quad \sum \frac{1}{n(n+1)}$

$\boxed{\frac{1}{n^2} \geq \frac{1}{n(n+1)}}$

$\boxed{\frac{1}{n^2} \leq \frac{100}{n(n+1)}}$??

$100 \cdot n^2 \geq n(n+1)$?? $\quad n \geq 1$

$\boxed{(10n)(10n) \geq n \cdot (n+1)}$

$\boxed{ES.3}$ $\sum \frac{1}{(\ln n)^{\ln n}}$

$\boxed{(\ln(n))^{\ln n}} = e^{\ln\left(\ln(n)^{\ln n}\right)} =$

$= \left(e^{\ln n}\right)^{\ln(\ln(n))} =$

$= \boxed{n^{\ln(\ln n)}}$

$\sum \frac{1}{n^{\ln(\ln n)}}$ CONV.

$\boxed{\frac{1}{n^{\ln(\ln n)}}} < \boxed{\frac{1}{n^2}}$ $\quad n > e^{e^2}$

$\boxed{ES.4}$ $\sum \frac{1}{(\ln n)^{\ln(\ln n)}}$

$(\ln n)^{\ln(\ln n)} = e^{\boxed{\ln(\ln n) \cdot \ln(\ln n)}} < e^{\boxed{\ln n}} = n$ (DEF. IN n)

$\boxed{\frac{1}{(\ln n)^{\ln(\ln n)}}} > \boxed{\frac{1}{n}}$

$(\ln(\ln n))^2 < \ln n$?

$(\ln x)^2 < x$?

$\ln x < \sqrt{x}$ SI

$\boxed{T.2}$ (CR. CONF. ASINT.)

DATE (a_n) E (b_n) A VALORI IN $(0, +\infty)$ E TALI CHE $\frac{a_n}{b_n} \to 1$ PER $n \to +\infty$. ALLORA

$\sum a_n$ E $\sum b_n$ HANNO ST. CARATTERE

$\boxed{\text{DIM}}$ POICHÉ $\frac{a_n}{b_n} \to 1$, DEF. IN n SI HA:

$$\frac{1}{2} < \frac{a_n}{b_n} < \frac{3}{2}$$

QUINDI, DEF IN n, SI HA

$\boxed{\frac{1}{2} b_n < a_n < \frac{3}{2} b_n}$

QUINDI SE $\sum a_n$ CONV. $\Rightarrow \sum \frac{1}{2} b_n$ CONV $\Rightarrow \sum b_n$

[...]

$\boxed{ES.5}$ $\sum_{n=1}^{+\infty} \left(1 - \cos \frac{1}{n}\right)$ CONV.

$1 - \cos x \approx \frac{x^2}{2}$ PER $x \to 0$

$\boxed{1 - \cos \frac{1}{n}} \approx \frac{1}{2n^2}$

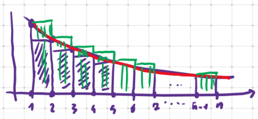
$\boxed{T.3}$ $\boxed{\text{CR. INTEGRALE}}$

DATA $f: [1, +\infty) \to \mathbb{R}$ DECRESCENTE E POSITIVA

ALLORA $\int_1^{+\infty} f(x)\,dx$ E $\sum_{n=1}^{+\infty} f(n)$

HANNO STESSO CARATTERE

$\boxed{\text{OSS.}}$



$S_{n-1} = \sum_{k=1}^{n-1} f(k) \geq \boxed{\int_1^n f(x)\,dx} \geq \sum_{k=2}^{n} f(k) = \boxed{-f(1) + S_n}$

$\boxed{\lim_{c\to+\infty} \int_1^c f(x)\,dx} = \boxed{\lim_{n\to+\infty} \int_1^n f(x)\,dx}$

$\sum f(n)$ CONV $\Rightarrow S_n \to \lambda \in \mathbb{R} \Rightarrow S_{n-1} \to \lambda \in \mathbb{R} \Rightarrow \int_1^n f(x)\,dx \to$ $\boxed{\text{lim finito}}$

$\sum f(n)$ DIV. $\Rightarrow S_n \to +\infty \Rightarrow \int_1^n f(x)\,dx \to +\infty$

$\boxed{APP.1}$

$f(x) = \frac{1}{x^\alpha}$ $\quad \int_1^{+\infty} \frac{1}{x^\alpha}$ $\quad \sum \frac{1}{n^\alpha}$

$g(x) = \boxed{\frac{1}{x^\alpha (\ln x)^\beta}}$

$\frac{(\ln x)^b}{x^a}$

$\left(\frac{\ln x}{x^{\frac{a}{b}}}\right)^b$

$\frac{1}{\gamma} \frac{\ln x^\gamma}{x^\gamma}$ $\quad \boxed{\frac{\ln y}{y}} = h(y)$

$h'(y) = \left(\frac{1}{y} \cdot \ln y\right)' = -\frac{1}{y^2} \cdot \ln y + \frac{1}{y^2} =$

$= \frac{1}{y^2}(1 - \ln y)$

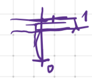
$\boxed{ES.7}$ $\sum_{n=1}^{+\infty} \frac{1}{\ln(n!)}$

$\boxed{\frac{1}{\ln(n!)}} \geq \frac{1}{\ln(n^n)} = \boxed{\frac{1}{n \ln n}}$

$\frac{1}{\ln(n^n)} = \frac{1}{n \ln n}$

$\boxed{T.4}$ (CR. RADICE)

DATA (a_n) A TERMINI POSITIVI SI HA:



→ 1) $\exists \lambda \in [0,1)$ t.c. $\boxed{\sqrt[n]{a_n} \leq \lambda}$ (**) def. in n ALLORA $\sum a_n$ CONV.

→ 2) FREQ IN n $\underbrace{\sqrt[n]{a_n} \geq 1}_{(*)}$ $\Rightarrow \sum a_n$ DIV

$\boxed{\text{DIM}}$

(*) $\Leftrightarrow a_n \geq 1$ FREQ. $\Rightarrow a_n \not\to 0 \Rightarrow \sum a_n$ DIV.

(**) $\Leftrightarrow \boxed{a_n \leq \lambda^n} = \dots$

$\boxed{T}$

1) $\boxed{\frac{a_{n+1}}{a_n} \leq \lambda}$

2) DEF IN n $\boxed{\frac{a_{n+1}}{a_n} \geq 1}$ $\Rightarrow \sum a_n$ divergente

$\boxed{a_{n+1} \geq a_n}$

n_0

a_{n_0}

$a_1 \leq \lambda a_0$

$a_2 \leq \lambda a_1 \leq \lambda^2 a_0$

$\vdots$

$a_n \leq \dots \leq \lambda^n a_0$

$a_{n+1} \leq \lambda \cdot a_n \leq \lambda \cdot \lambda^n a_0 = \lambda^{n+1} a_0$