Analisi Matematica 1- Lista n. 30

Studio del carattere di Serie a Termini Positivi

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Studiare il carattere delle seguenti serie:

1) $\sum_{n=1}^{+\infty} \frac{1}{n^4+1}$ 2) $\sum_{n=0}^{+\infty} \frac{1}{2^n+3^n}$

3) $\sum_{n=1}^{+\infty} \frac{n^4}{n^7+n^5}$ 4) $\sum_{n=1}^{+\infty} \ln\left(1+\frac{1}{n}\right)$

5) $\sum_{n=0}^{+\infty} e^{-\sqrt{n}}$ 6) $\sum_{n=1}^{+\infty} \frac{1}{n^n}$

7) $\sum_{n=0}^{+\infty} \frac{e^{\sin n}}{1+n^2}$ 8) $\sum_{n=1}^{+\infty} \frac{1}{n^{\sqrt{n}}}$

9) $\sum_{n=1}^{+\infty} \frac{e^{\cos n}}{n}$ 10) $\sum_{n=1}^{+\infty} \ln\left(1+\frac{1}{n}\right)^{\sqrt{n}}$

11) $\sum_{n=1}^{+\infty} \frac{1}{(1+n)\cdot\sqrt{\ln(1+n^2)}}$ 12) $\sum_{n=1}^{+\infty} \frac{1}{n\cdot\ln^2(1+\sqrt{n})}$

13) $\sum_{n=0}^{+\infty} \frac{(n!)^2}{(2n)!}$ 14) $\sum_{n=1}^{+\infty} \frac{(3n)!}{n^{3n}}$

15) $\sum_{n=1}^{+\infty} \frac{(n+1)^{n-1}}{n^{n+1}}$ 16) $\sum_{n=3}^{+\infty} \left(\frac{\ln(\ln n)}{\ln n}\right)^{\sqrt{n}}$

17) $\sum_{n=2}^{+\infty} \frac{1}{\ln(n!)}$ 18) $\sum_{n=2}^{+\infty} \frac{1}{(\ln n)^{\ln n}}$

19) $\sum_{n=3}^{+\infty} \frac{1}{(\ln(\ln n))^{\ln n}}$ 20) $\sum_{n=3}^{+\infty} \frac{1}{(\ln n)^{\ln(\ln n)}}$

21) $\sum_{n=3}^{+\infty} n^{-1-\frac{1}{\ln(\ln n)}}$ 22) $\sum_{n=1}^{+\infty} n^{-\sqrt{n}}$

23) $\sum_{n=1}^{+\infty} \frac{\cos(\cos n)}{n\cdot\ln n}$ 24) $\sum_{n=1}^{+\infty} (\sin(\sin n))^n$

25) $\sum_{n=1}^{+\infty}\left(\arctan\frac{1}{\sqrt[4]{n}}-\sin\frac{1}{\sqrt[4]{n}}\right)$ 26) $\sum_{n=1}^{+\infty}\left(\sqrt{1-\frac{1}{\sqrt{n}}}-\cos\frac{1}{\sqrt[4]{n}}\right)$

27) $\sum_{n=1}^{+\infty} \frac{\sqrt{n!}}{(\sqrt{n})^n}$ 28) $\sum_{n=1}^{+\infty} \frac{1}{\sqrt[n]{(n!)^2}}$

29) $\sum_{n=1}^{+\infty} \frac{(n!)!}{n^{n^n}}$ 30) $\sum_{n=1}^{+\infty} \frac{(n!)^{n!}}{(n^n)!}$

31) $\sum_{n=1}^{+\infty} \frac{1+\cos n}{n}$ 32) $\sum_{n=1}^{+\infty} \left(e-\sum_{k=0}^{n}\frac{1}{k!}\right)$

33) $\sum_{n=1}^{+\infty}\left(\sum_{k=n}^{+\infty}\frac{1}{k^k}\right)$ 34) $\sum_{n=1}^{+\infty}\left(\sum_{k=n}^{+\infty}\frac{1}{k^2}\right)$

Studiare, al variare del parametro reale $a>0$, il carattere delle seguenti serie:

35) $\sum_{n=1}^{+\infty}\left(1-\cos\frac{1}{n}\right)^a$ 36) $\sum_{n=1}^{+\infty}\left(\frac{\pi}{2}-\arctan(n^a)\right)$

37) $\sum_{n=1}^{+\infty}\left(\ln\left(\cos\frac{1}{\sqrt[4]{n}}\right)+\frac{1}{2\sqrt{n}}+\frac{a}{n}\right)$ 38) $\sum_{n=1}^{+\infty}\left(\cos\left(\sin\frac{1}{\sqrt[4]{n}}\right)-1+\frac{1}{2\sqrt{n}}-\frac{a}{n}\right)$

39) $\sum_{n=1}^{+\infty}\frac{1}{a^{\ln n}}$ 40) $\sum_{n=2}^{+\infty}\left(\frac{\log_{2^n}4}{\log_4 n^n}\right)^a$

41) $\sum_{n=1}^{+\infty}\left(1-\frac{1}{n^3}\right)^{n^a}$ 42) $\sum_{n=1}^{+\infty}\frac{n!}{n^n}a^n$

43) $\sum_{n=3}^{+\infty}\frac{1}{n\cdot\ln n\cdot(\ln(\ln n))^a}$ 44) $\sum_{n=1}^{+\infty}\left(e-\left(1+\frac{1}{n}\right)^n\right)^a$

45) $\sum_{n=1}^{+\infty}\left(e-\sum_{k=0}^{n}\frac{1}{k!}\right)^a$ 46) $\sum_{n=1}^{+\infty}\frac{1}{\sqrt[n]{(n!)^a}}$

47) $\sum_{n=1}^{+\infty}\frac{1}{n^2}\left(\frac{n+2}{n+1}\right)^{an\ln n}$ 48) $\sum_{n=1}^{+\infty}\left(\frac{n^2+n-1}{n^2+3n+5}\right)^{n^a}$

49) $\sum_{n=1}^{+\infty}\frac{n!\cdot n^{n+1}}{(2n)!}a^n$ 50) $\sum_{n=1}^{+\infty}\frac{n^{2n}}{(2n)!}a^n$

Analisi Matematica 2 - 6 Aprile 2022 - docente: Callegari

Lezione 15

Serie a termini positivi (II)

(... CONTINUA DA LEZ. SCORSA)

1) CRITERIO DEL RAPPORTO.

2) ESEMPI: 2A $\sum_{n=1}^{+\infty}\left(1-\frac{1}{n}\right)^{n^2}$ 2B $\sum_{n=1}^{+\infty}\frac{n!}{n^n}A^n$ 2C $\sum_{n=1}^{+\infty}\frac{n!\cdot n^{n+1}}{(2n)!}A^n$

2D $\sum_{n=1}^{+\infty} a_n$ DOVE a_n È DEFINITA DA $\begin{cases} a_{n+1}=\frac{1}{2}\sin a_n \\ a_0=1\end{cases}$

2E $\sum_{n=1}^{+\infty}\frac{n^{2n}}{(2n)!}A^n$

3) CRITERIO DEL CONFRONTO DI RAPPORTI

4) CRITERIO DI GAUSS

5) ULTERIORI ESEMPI PRESI DA EXE ON LINE

T.1 (CR. RAPPORTO)

DATA (a_n) A TERMINI IN $(0,+\infty)$ SI HA:

1) $\left(\exists\lambda\in(0,1)\ \text{t.c. DEF. IN } n\ \frac{a_{n+1}}{a_n}\le\lambda\right)\Rightarrow\sum a_n$ CONV.

2) $\left(\text{DEF. IN } n\ \frac{a_{n+1}}{a_n}\ge 1\right)\Rightarrow\sum a_n$ DIVERGE

DIM ① POSSO SUPPORRE CHE $\frac{a_{n+1}}{a_n}\le\lambda$ VALGA $\forall n\in\mathbb{N}$ PERCHÉ POSSO CAMBIARE UN NUMERO FINITO DI TERMINI DI (a_n) SENZA CHE CAMBI IL CAR.

$a_1\le\lambda a_0$

$a_2\le\lambda a_1\le\lambda^2 a_0$

$\vdots$

$a_n\le\lambda^n a_0$

$a_{n+1}\le\lambda a_n\le\lambda\cdot\lambda^n a_0=\lambda^{n+1}a_0$

$\forall n\in\mathbb{N}\quad \boxed{a_n}\le\boxed{a_0\cdot\lambda^n}$

$\boxed{\sum_{n=0}^{+\infty}a_0\cdot\lambda^n}=\boxed{a_0\sum_{n=0}^{+\infty}\lambda^n}$

(CR. CONF.) $a_n\ge\boxed{a_{n_0}}$ C

$\Downarrow$

$\sum_{n=0}^{+\infty}a_n$ CONV.

② $\exists n_0\in\mathbb{N}$ t.c. $\forall n\ge n_0$ $\boxed{\frac{a_{n+1}}{a_n}\ge 1}\Rightarrow\forall n\ge n_0$ $a_{n+1}\ge a_n$

$a_n\not\to 0$

$\Downarrow$

$\sum a_n$ NON CONVERGE

$\Downarrow$

$\sum a_n$ DIVERGE.

CR. RADICE $\Rightarrow$ CONV.

2A $\sum_{n=1}^{+\infty}\left(1-\frac{1}{n}\right)^{n^2}$

$\sqrt[n]{a_n}=\sqrt[n]{\left(1-\frac{1}{n}\right)^{n^2}}=\left(1-\frac{1}{n}\right)^n\to\frac{1}{e}$

~~$\sqrt[n]{a_n}\le\frac{1}{2}$ DEF. IN n~~

$0\le\left(1-\frac{1}{n}\right)^{n^2}=\left(\left(1-\frac{1}{n}\right)^n\right)^n\le\left(\frac{1}{2}\right)^n$ DEF. IN n, CONV.

2B $\sum_{n=1}^{+\infty}\frac{n!}{n^n}A^n$ $(A>0)$

$\frac{a_{n+1}}{a_n}=\frac{(n+1)!}{(n+1)^{n+1}}\cdot A^{n+1}\cdot\frac{n^n}{n!}\cdot\frac{1}{A^n}$

$=\frac{\cancel{(n+1)}}{\cancel{(n+1)}}\cdot\frac{n^n}{(n+1)^n}\cdot A=\frac{A}{\left(\frac{n+1}{n}\right)^n}=\frac{A}{\left(1+\frac{1}{n}\right)^n}\to\frac{A}{e}$

SE $0<A<e$ CONV.

SE $A>e$ DIV.

SE $A=e$ $\frac{a_{n+1}}{a_n}\to 1$ PERÒ: PERCHÉ $\left(1+\frac{1}{n}\right)^n$ CRESCENDO

$\underbrace{\frac{a_{n+1}}{a_n}=\frac{e}{\left(1+\frac{1}{n}\right)^n}\ge\frac{e}{e}=1}$

$\Downarrow$

$\sum a_n$ DIV.

$\sum_{n=1}^{+\infty}\frac{n!\cdot n^{n+1}}{(2n)!}A^n$ $(A>0)$

$\frac{a_{n+1}}{a_n}=\frac{(n+1)!\cdot(n+1)^{n+2}}{(2n+2)!}\cdot A^{n+1}\cdot\frac{(2n)!}{n!\cdot n^{n+1}}\cdot\frac{1}{A^n}=$

$=\frac{\cancel{(n+1)}(n+1)}{2\cancel{(n+1)}(2n+1)}\cdot\frac{(n+1)^{n+1}}{n^{n+1}}\cdot A=$

$=\frac{A}{4}\cdot\frac{2n+2}{2n+1}\cdot\left(1+\frac{1}{n}\right)^{n+1}\longrightarrow\frac{A}{4}\cdot e$

SE $A>\frac{4}{e}\Rightarrow\sum a_n$ DIVERGE

SE $0<A<\frac{4}{e}\Rightarrow\sum a_n$ CONV.

SE $A=\frac{4}{e}$

$\frac{4}{e}\cdot\frac{1}{4}\cdot\boxed{\frac{2n+2}{2n+1}}\cdot\overbrace{\boxed{\left(1+\frac{1}{n}\right)^{n+1}}}^{\searrow e}>1$

2E $\sum_{n=1}^{+\infty}\frac{n^{2n}}{(2n)!}A^n$

$\frac{a_{n+1}}{a_n}=\frac{(n+1)^{2n+2}}{(2n+2)!}\cdot A^{n+1}\cdot\frac{(2n)!}{n^{2n}}\cdot\frac{1}{A^n}=$

$=A\cdot\frac{(n+1)^2}{2(2n+1)(2n+2)}\cdot\frac{(n+1)^{2n}}{n^{2n}}=\frac{A}{4}\cdot\frac{2n+2}{2n+1}\left(\left(1+\frac{1}{n}\right)^n\right)^2\longrightarrow\frac{e^2\cdot A}{4}$

$A>\frac{4}{e^2}\Rightarrow\sum a_n$ DIV.

$0<A<\frac{4}{e^2}\Rightarrow\sum a_n$ CONV.

$A=\frac{4}{e^2}$

$=\frac{2n+2}{2n+1}\cdot\frac{1}{e^2}\cdot\left(1+\frac{1}{n}\right)^{2n}=$

$=2n\cdot\left(\frac{1}{n}-\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right)\right)-2=-\frac{1}{n}+o\left(\frac{1}{n}\right)$

$=\left(1+\frac{1}{2n+1}\right)\cdot e^{2n\ln\left(1+\frac{1}{n}\right)-2}=$

$=\left(1+\frac{1}{2n+1}\right)\cdot\left(1+\left(-\frac{1}{n}+o\left(\frac{1}{n}\right)\right)+o\left(\frac{1}{n}\right)\right)=$

$=\left(1+\frac{1}{2n+1}\right)\cdot\left(1-\frac{1}{n}+o\left(\frac{1}{n}\right)\right)=$

$=\left(1+\frac{1}{2n+1}-\frac{1}{n}+o\left(\frac{1}{n}\right)\right)=$

$=1+\frac{1}{2n+1}-\frac{1}{2n}+\frac{1}{2n}-\frac{1}{n}+o\left(\frac{1}{n}\right)=1-\frac{1}{2}\cdot\frac{1}{n}+o\left(\frac{1}{n}\right)$

$\frac{2n-2n-1}{(2n+1)2n}=-\frac{1}{2n(2n+1)}=o\left(\frac{1}{n}\right)$

$\frac{a_{n+1}}{a_n}=1-\frac{1}{2}\cdot\frac{1}{n}+o\left(\frac{1}{n}\right)<1$

T.3 (CR. GAUSS)

DATA (a_n) A TERMINI (STR.) POSITIVI

1) $\left(\frac{a_{n+1}}{a_n}=1-\frac{\alpha}{n}+o\left(\frac{1}{n}\right)\ \text{CON}\ \alpha>1\right)\Rightarrow\sum a_n$ CONV.

2) $\left(\frac{a_{n+1}}{a_n}=1-\frac{\beta}{n}+o\left(\frac{1}{n}\right)\ \text{CON}\ 0<\beta<1\right)\Rightarrow\sum a_n$ DIV.

T.2 (CR. CONF. DI RAPPORTI)

DATE (a_n) E (b_n) A TERMINI (STR.) POS.

TALI CHE DEF. IN n $\frac{a_{n+1}}{a_n}\ge\frac{b_{n+1}}{b_n}$ ALLORA

1) $\sum a_n$ (*)

2)

CONV. $\Rightarrow\sum b_n$ CONV.

$\sum b_n$ DIV $\Rightarrow\sum a_n$ DIV.

DIM. (SI PUÒ SUPPORRE CHE (*) VALGA $\forall n$)

$\frac{a_n}{a_0}=\frac{a_n}{a_{n-1}}\cdot\frac{a_{n-1}}{a_{n-2}}\cdot\frac{a_{n-2}}{a_{n-3}}\cdot\ldots\cdot\frac{a_2}{a_1}\cdot\frac{a_1}{a_0}\ge\frac{b_n}{b_{n-1}}\cdot\frac{b_{n-1}}{b_{n-2}}\cdot\frac{b_{n-2}}{b_{n-3}}\cdot\ldots\cdot\frac{b_2}{b_1}\cdot\frac{b_1}{b_0}=\frac{b_n}{b_0}$

$\frac{a_n}{a_0}\ge\frac{b_n}{b_0}\quad\forall n$

$a_n\ge\boxed{\frac{a_0}{b_0}}\cdot b_n\Rightarrow\left(\sum a_n\ \text{CONV.}\Rightarrow\sum b_n\ \text{CONV.}\right)$

(CR. CONF.)

$\left(b_n=\frac{1}{n^\alpha}\right)$ $\sum b_n$ $\sum\frac{1}{n^\alpha}$

$\frac{b_{n+1}}{b_n}=\frac{\frac{1}{(n+1)^\alpha}}{\frac{1}{n^\alpha}}=\left(\frac{n+1}{n}\right)^{-\alpha}=\left(1+\frac{1}{n}\right)^{-\alpha}=\boxed{1-\frac{\alpha}{n}}+o\left(\frac{1}{n}\right)$

$\frac{a_{n+1}}{a_n}=1-\frac{\alpha}{n}+o\left(\frac{1}{n}\right)$