

Lezione 15

Serie a termini positivi (II)

(... CONTINUA DA LEZ. SCORSA)

- 1) CRITERIO DEL RAPPORTO
- 2) ESEMPLI:
 - ZB $\sum_{n=1}^{\infty} (1-\frac{1}{n})^n$
 - ZB $\sum_{n=1}^{\infty} \frac{n!}{n^n} A^n$
 - ZB $\sum_{n=1}^{\infty} \frac{n! \cdot n^{n^2}}{(2n)!} A^n$
 - ZD $\sum_{n=1}^{\infty} a_n$ DOVE a_n È DEFINITO DA $\begin{cases} a_{n+1} = \frac{1}{2} a_n \\ a_1 = 1 \end{cases}$
 - ZE $\sum_{n=1}^{\infty} \frac{1}{(n!)^2} A^n$
- 3) CRITERIO DEL CONFRONTO DI RAPPORTI
- 4) CRITERIO DI GAUSS
- 5) ULTERIORI ESEMPLI PRESI DA EYE OF LEMMA

T.1 (CR. RAPPORTO)

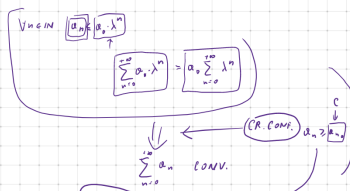
DATI (a_n) A TERMINI IN $(0, \infty)$ SI HA:

- 1) $(\exists \lambda \in (0, 1) \ \exists c \text{ DEF. IN } n \ \frac{a_{n+1}}{a_n} < \lambda) \Rightarrow \sum a_n \text{ CONV.}$
- 2) $(\text{DEF. IN } n \ \frac{a_{n+1}}{a_n} > 1) \Rightarrow \sum a_n \text{ DIVERGENTE}$

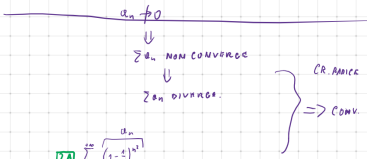
Dim 1) POSSO SUPPORRE CHE $\frac{a_{n+1}}{a_n} < \lambda$ A PARTIRE DA UN CERTO INDICE n_0

POSSO CAMBIARE UN NUMERO FINITO DI TERMINI DI (a_n) SENZA CHE CAMBI IL CAR.

$u_n \in \lambda a_n$
 $a_n > \lambda a_n < \lambda^2 a_n$
 \vdots
 $a_n < \lambda^n a_0$
 $a_{n+1} < \lambda a_n < \lambda^2 a_0 = \lambda^{n+1} a_0$



1) $\exists n_0 \in \mathbb{N}$ t.c. $\forall n \geq n_0 \ \frac{a_{n+1}}{a_n} < \lambda < 1 \Rightarrow \sum_{n=n_0}^{\infty} a_n < \sum_{n=n_0}^{\infty} \lambda^n a_{n_0} < \sum_{n=0}^{\infty} \lambda^n < \infty$



ZB $\sum_{n=1}^{\infty} \frac{a_n}{(1-\frac{1}{n})^n}$

$\sqrt[n]{a_n} = \sqrt[n]{(1-\frac{1}{n})^{n^2}} = (1-\frac{1}{n})^n \rightarrow \frac{1}{e} < 1$

$0 < (1-\frac{1}{n})^n = (1-\frac{1}{n})^n \leq (\frac{1}{2})^n$

ZB $\sum_{n=1}^{\infty} \frac{a_n}{n!} A^n$

$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot A^{n+1} \cdot \frac{n!}{n!} \cdot \frac{1}{A^n}$

$= \frac{(n+1)}{(n+1)^{n+1}} \cdot A = \frac{A}{(1+\frac{1}{n})^{n+1}} \rightarrow \frac{A}{e}$

SE $0 < \frac{A}{e} < 1 \Rightarrow \sum a_n \text{ CONV.}$

SE $A > e \Rightarrow \sum a_n \text{ DIVERG.}$

SE $A = e$

$\frac{a_{n+1}}{a_n} \rightarrow 1$ PERÒ: $\frac{a_{n+1}}{a_n} = \frac{e}{(1+\frac{1}{n})^{n+1}} > 1$ PERCHÉ $(1+\frac{1}{n})^{n+1} < e$ CRESCENDO

$\Rightarrow \sum a_n \text{ DIVERG.}$

$\sum_{n=1}^{\infty} \frac{a_n}{(2n)!} A^n$

$\frac{a_{n+1}}{a_n} = \frac{(n+1)! (n+1)^{2n+2}}{(2n+2)!} \cdot A^{n+1} \cdot \frac{(2n)!}{n! \cdot n^{2n}} \cdot \frac{1}{A^n}$

$= \frac{(n+1)! (n+1)^{2n+2}}{(2n+2)(2n+1)(2n)!} \cdot \frac{(n+1)^{2n+2}}{n^{2n}} \cdot A = \frac{A}{2} \cdot \frac{(n+1)^2}{n^2} \cdot (1+\frac{1}{n})^{2n+2} \rightarrow \frac{A}{2} e$

SE $A > \frac{e}{2} \Rightarrow \sum a_n \text{ DIVERG.}$

SE $0 < A < \frac{e}{2} \Rightarrow \sum a_n \text{ CONV.}$

SE $A = \frac{e}{2}$

$\frac{a_{n+1}}{a_n} = \frac{e}{2} \cdot \frac{(n+1)^2}{n^2} \cdot (1+\frac{1}{n})^{2n+2} > 1$

ZB $\sum_{n=1}^{\infty} \frac{n^{n^2}}{(n!)^2} A^n$

$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{(n+1)^2}}{(2n+2)!} \cdot A^{n+1} \cdot \frac{(n!)^2}{n^{2n}} \cdot \frac{1}{A^n}$

$= A \cdot \frac{(n+1)^{2n+2}}{(2n+2)(2n+1)(n!)^2} \cdot \frac{(n!)^2}{n^{2n}} = \frac{A}{2} \cdot \frac{(n+1)^{2n+2}}{(2n+1)n^{2n}} \rightarrow \frac{e^2 A}{4}$

$A > \frac{4}{e^2} \Rightarrow \sum a_n \text{ DIVERG.}$

$0 < A < \frac{4}{e^2} \Rightarrow \sum a_n \text{ CONV.}$

$A = \frac{4}{e^2}$

$= \frac{2n+2}{2n+1} \cdot \frac{(1+\frac{1}{n})^{2n+2}}{n^2} = \frac{2n+2}{2n+1} \cdot \frac{1}{n^2} \cdot (1+\frac{1}{n})^{2n+2}$

$= (1+\frac{1}{2n+1}) \cdot (1+\frac{1}{n}) \cdot (1+\frac{1}{n})^{2n} = (1+\frac{1}{2n+1}) \cdot (1+\frac{1}{n}) \cdot e^2$

$= (1+\frac{1}{2n+1}) \cdot (1+\frac{1}{n}) \cdot e^2 > e^2$

$= (1+\frac{1}{2n+1}) \cdot (1+\frac{1}{n}) \cdot e^2 > e^2 > 4$

$\Rightarrow \sum a_n \text{ DIVERG.}$

T.3 (CR. GAUSS)

DATI (a_n) A TERMINI POSITIVI:

- 1) $(\frac{a_{n+1}}{a_n} = 1 - \frac{\alpha}{n} + o(\frac{1}{n})) \text{ CON } \alpha > 1 \Rightarrow \sum a_n \text{ CONV.}$
- 2) $(\frac{a_{n+1}}{a_n} = 1 - \frac{\alpha}{n} + o(\frac{1}{n})) \text{ CON } 0 < \alpha < 1 \Rightarrow \sum a_n \text{ DIVERG.}$

T.2 (CR. CRIF. DI RAPPORTI)

DATI $(a_n) = (b_n)$ A TERMINI POS.

TALI CHE DEF. IN $n \ \frac{a_{n+1}}{a_n} \geq \frac{b_{n+1}}{b_n}$ ALL'INFI.

- 1) $\sum a_n$
- 2) $(\sum b_n)$

CR. CONV. $\Rightarrow \sum b_n \text{ CONV.}$

$\sum b_n \text{ DIVERG.} \Rightarrow \sum a_n \text{ DIVERG.}$

Dim 1) POSSO SUPPORRE CHE (SI VALGA V)

$\frac{a_n}{a_{n-1}} = \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdot \frac{a_{n-2}}{a_{n-3}} \cdot \dots \cdot \frac{a_2}{a_1} \geq \frac{b_n}{b_{n-1}} \geq \frac{b_{n-1}}{b_{n-2}} \geq \dots \geq \frac{b_2}{b_1} = \frac{b_n}{b_1}$

$\frac{a_n}{a_{n-1}} \geq \frac{b_n}{b_{n-1}} \ \forall n$

$a_n \geq \frac{b_n}{b_{n-1}} \cdot b_{n-1} \Rightarrow \sum a_n \text{ CONV.} \Rightarrow \sum b_n \text{ CONV.}$

$b_n = \frac{1}{n}$

$\frac{b_{n+1}}{b_n} = \frac{1}{n+1} \cdot n = \frac{n}{n+1} = (1-\frac{1}{n+1})^n = (1-\frac{\alpha}{n+1})^n < 1$

$\frac{a_{n+1}}{a_n} = 1 - \frac{\alpha}{n} + o(\frac{1}{n})$