Analisi Matematica 2 - 13 Aprile 2022 - docente: Callegari

Lezione 18

Esercizi sulle Serie

15 L.30 $\sum_{n=1}^{+\infty} \frac{(n+1)^{n-1}}{n^{n+1}}$ CONV. PER CR. CONFRONTO

$\frac{(n+1)^{n-1}}{n^{n+1}} = \boxed{\frac{(n+1)^n}{n^n}} \cdot \frac{1}{n\cdot(n+1)} = \underbrace{\boxed{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{n(n+1)}}_{a_n} \le e \cdot \boxed{\frac{1}{n(n+1)}}$

16 L.30 $\sum_{n=3}^{+\infty} \left(\boxed{\frac{\ln(\ln(n))}{\ln n}}\right)^{\sqrt{n}}$ CONV. PER CONFRONTO CON $\sum \frac{1}{n^2}$

$\downarrow$ 0 ?

$\underbrace{\boxed{\ln(\ln n)}}_{\ln(a_n)} \ll \underbrace{\boxed{\ln(n)}}_{a_n}$

$2^{\sqrt{n}} \gg (\sqrt{n})^\alpha$ $\boxed{\alpha=4}$ $2^{\sqrt{n}} \gg n^2$

DEF. IN VIA

$0 < \left(\frac{\ln(\ln n)}{\ln n}\right)^{\sqrt{n}} < \left(\frac{1}{2}\right)^{\sqrt{n}} < \boxed{\frac{1}{n^2}}$ Σ CONVERGE

Analisi Matematica 1- Lista n. 30

Studio del carattere di Serie a Termini Positivi

www.problemisvolti.it

Studiare il carattere delle seguenti serie:

1) $\sum_{n=1}^{+\infty} \frac{1}{n^4+1}$ 2) $\sum_{n=0}^{+\infty} \frac{1}{2^n+3^n}$

3) $\sum_{n=1}^{+\infty} \frac{n^4}{n^2+n^4}$ 4) $\sum_{n=1}^{+\infty} \ln\left(1+\frac{1}{n}\right)$

5) $\sum_{n=0}^{+\infty} e^{-\sqrt{n}}$ 6) $\sum_{n=1}^{+\infty} \frac{1}{n^n}$

7) $\sum_{n=0}^{+\infty} \frac{e^{\sin n}}{1+n^2}$ 8) $\sum_{n=1}^{+\infty} \frac{1}{n^{\ln n}}$

9) $\sum_{n=1}^{+\infty} \frac{e^{\cos n}}{n}$ 10) $\sum_{n=1}^{+\infty} \ln\left(1+\frac{1}{n^2}\right)^{\sqrt{n}}$

11) $\sum_{n=1}^{+\infty} \frac{1}{(1+n)\cdot\sqrt{\ln(1+n^4)}}$ 12) $\sum_{n=1}^{+\infty} \frac{1}{n\cdot\ln^2(1+\sqrt{n})}$

13) $\sum_{n=0}^{+\infty} \frac{(n!)^2}{(2n)!}$ 14) $\sum_{n=1}^{+\infty} \frac{(3n)!}{n^{4n}}$

15) $\sum_{n=1}^{+\infty} \frac{(n+1)^{n-1}}{n^{n+1}}$ 16) $\sum_{n=3}^{+\infty} \left(\frac{\ln(\ln n)}{\ln n}\right)^{\sqrt{n}}$

17) $\sum_{n=2}^{+\infty} \frac{1}{\ln(n!)}$ 18) $\sum_{n=2}^{+\infty} \frac{1}{(\ln n)^{\ln n}}$

19) $\sum_{n=3}^{+\infty} \frac{1}{(\ln(\ln n))^{\ln n}}$ 20) $\sum_{n=3}^{+\infty} \frac{1}{(\ln n)^{\ln(\ln n)}}$

21) $\sum_{n=3}^{+\infty} n^{-1-\frac{1}{\ln(\ln n)}}$ 22) $\sum_{n=1}^{+\infty} n^{-\sqrt[n]{n}}$

23) $\sum_{n=1}^{+\infty} \frac{\cos(\cos n)}{n\cdot\ln n}$ ($n=3$) 24) $\sum_{n=1}^{+\infty} (\sin(\sin n))^n$

25) $\sum_{n=1}^{+\infty} \left(\arctan\frac{1}{\sqrt[4]{n}} - \sin\frac{1}{\sqrt[4]{n}}\right)$ 26) $\sum_{n=1}^{+\infty} \left(\sqrt{1-\frac{1}{\sqrt{n}}} - \cos\frac{1}{\sqrt[4]{n}}\right)$

27) $\sum_{n=1}^{+\infty} \frac{\sqrt{n!}}{(\sqrt{n})^n}$ 28) $\sum_{n=1}^{+\infty} \frac{1}{\sqrt[n]{(n!)^2}}$

29) $\sum_{n=1}^{+\infty} \frac{(n!)!}{n^{n^n}}$ 30) $\sum_{n=1}^{+\infty} \frac{(n!)^{n!}}{(n^n)!}$

31) $\sum_{n=1}^{+\infty} \frac{1+\cos n}{n}$ $\boxed{32) \sum_{n=1}^{+\infty} \left(e - \sum_{k=0}^{n} \frac{1}{k!}\right)}$

33) $\sum_{n=1}^{+\infty} \left(\sum_{k=n}^{+\infty} \frac{1}{k^k}\right)$ 34) $\sum_{n=1}^{+\infty} \left(\sum_{k=n}^{+\infty} \frac{1}{k^2}\right)$

Studiare, al variare del parametro reale $a>0$, il carattere delle seguenti serie:

35) $\sum_{n=1}^{+\infty} \left(1-\cos\frac{1}{n}\right)^a$ 36) $\sum_{n=1}^{+\infty} \left(\frac{\pi}{2} - \arctan(n^a)\right)$

37) $\sum_{n=1}^{+\infty} \left(\ln\left(\cos\frac{1}{\sqrt{n}}\right) + \frac{1}{2\sqrt{n}} + \frac{a}{n}\right)$ 38) $\sum_{n=1}^{+\infty} \left(\cos\left(\sin\frac{1}{\sqrt{n}}\right) - 1 + \frac{1}{2\sqrt{n}} - \frac{a}{n}\right)$

39) $\sum_{n=1}^{+\infty} \frac{1}{a^{\ln n}}$ 40) $\sum_{n=2}^{+\infty} \left(\frac{\log_2 4}{\log_4 n^n}\right)^a$

41) $\sum_{n=1}^{+\infty} \left(1-\frac{1}{n^3}\right)^{n^a}$ 42) $\sum_{n=1}^{+\infty} \frac{n!}{n^n} a^n$

43) $\sum_{n=3}^{+\infty} \frac{1}{n\cdot\ln n\cdot(\ln(\ln n))^a}$ 44) $\sum_{n=1}^{+\infty} \left(e - \left(1+\frac{1}{n}\right)^n\right)^a$

45) $\sum_{n=1}^{+\infty} \left(e - \sum_{k=0}^{n} \frac{1}{k!}\right)^a$ 46) $\sum_{n=1}^{+\infty} \frac{1}{\sqrt[n]{(n!)^a}}$

47) $\sum_{n=1}^{+\infty} \frac{1}{n^2}\left(\frac{n+2}{n+1}\right)^{an\ln n}$ 48) $\sum_{n=1}^{+\infty} \left(\frac{n^2+n-1}{n^2+3n+5}\right)^{n^a}$

49) $\sum_{n=1}^{+\infty} \frac{n!\cdot n^{n+1}}{(2n)!} a^n$ 50) $\sum_{n=1}^{+\infty} \frac{n^{2n}}{(2n)!} a^n$

Analisi Matematica 1- Lista n. 31

Studio del carattere di Serie a Termini di segno qualsiasi

www.problemisvolti.it

Studiare convergenza semplice e assoluta delle seguenti serie:

1) $\sum_{n=1}^{+\infty} (-1)^n \tan\frac{1}{n}$ 2) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n+\sin n}$

3) $\sum_{n=2}^{+\infty} (-1)^n \left(1-\frac{1}{n}\right)^{n\ln n}$ 4) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{2n+\sin n}$

5) $\sum_{n=1}^{+\infty} (-1)^n \frac{n^2+4n+8}{n^3+2n^2+7n}$ 6) $\sum_{n=1}^{+\infty} \frac{\cos(\frac{\pi}{2}n)}{n}$

7) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{\ln n + e^{-n}}$ 8) $\sum_{n=1}^{+\infty} \sin\left(\pi n + \frac{1}{n^2}\right)$

9) $\sum_{n=1}^{+\infty} \frac{(-1)^n n}{n\ln n+1}$ 10) $\sum_{n=1}^{+\infty} \left(e^{\frac{(-1)^n}{\sqrt{n}}} - 1\right)$

11) $\sum_{n=1}^{+\infty} \ln\left(1+\frac{(-1)^{n+1}}{\sqrt[3]{n}}\right)$ 12) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{2n+(-1)^n n}$

13) $\sum_{n=1}^{+\infty} \sin\left(\pi n^2 + \frac{1}{n}\right)$ 14) $\sum_{n=1}^{+\infty} \sin\left(\pi n! + \frac{1}{n}\right)$

15) $\sum_{n=1}^{+\infty} \frac{\sqrt{2}\cdot\sin\left((2n+1)\frac{\pi}{4}\right)}{n^{2+(-1)^n}}$ 16) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{\sqrt{n}+\sin n}$

17) $\sum_{n=2}^{+\infty} \ln\left(\frac{n^2+2\cdot(-1)^n}{n^2+1}\right)$ 18) $\sum_{n=2}^{+\infty} \ln\left(\frac{n+2\cdot(-1)^n}{n+1}\right)$

19) $\sum_{n=1}^{+\infty} (-1)^n \left(e^{-\frac{2}{\sqrt{n}}} - \cos\frac{2}{\sqrt[4]{n}}\right)$ 20) $\sum_{n=1}^{+\infty} \left(\sin\frac{(-1)^n}{\sqrt[5]{n}} - \frac{(-1)^n}{\sqrt[5]{n}}\cdot\cos\frac{(-1)^n}{\sqrt{3}\cdot\sqrt[5]{n}}\right)$

21) $\sum_{n=1}^{+\infty} (-1)^n \left(2\cdot e^{\frac{1}{\sqrt{n}}} - 2\cdot\cos\frac{1}{\sqrt{n}} + \ln\left(1-\frac{2}{\sqrt{n}}\right)\right)$ 22) $\sum_{n=1}^{+\infty} \left(2\cdot e^{\frac{(-1)^n}{\sqrt[3]{n}}} - 2\cdot\cos\frac{(-1)^n}{\sqrt[3]{n}} + \ln\left(1-2\cdot\frac{(-1)^n}{\sqrt[3]{n}}\right)\right)$

23) $\sum_{n=1}^{+\infty} \frac{\sin(\pi\cdot\log_3 n)}{n}$ 24) $\sum_{n=1}^{+\infty} \frac{\sin n}{n}$

Studiare, al variare di $a>0$, convergenza semplice e assoluta delle seguenti serie:

25) $\sum_{n=1}^{+\infty} (-1)^n \left(\frac{1}{n} - \arctan\frac{1}{n}\right)^a$ 26) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^a+1}\left(1+\frac{1}{n}\right)^n$

27) $\sum_{n=2}^{+\infty} \frac{(-1)^n}{n\cdot(\ln n)^a}\left(1+\frac{1}{n}\right)^n$ 28) $\sum_{n=2}^{+\infty} (-1)^n \frac{(n+1)^{an}}{n^n\cdot\ln n}$

29) $\sum_{n=1}^{+\infty} (-1)^n n^{an} \sin\frac{1}{(n+100)!}$ 30) $\sum_{n=1}^{+\infty} \frac{1}{1+\left(-1-\frac{1}{n}\right)^n\cdot n^a}$

31) $\sum_{n=\lfloor a\rfloor+1}^{+\infty} \frac{(-1)^n}{n+a\sin n}$ 32) $\sum_{n=1}^{+\infty} \left(\sqrt{1+\frac{(-1)^n}{n^a}} - 1\right)$

33) $\sum_{n=2}^{+\infty} \frac{(-1)^n}{n^a+\cos(n\pi)}$ 34) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^a+\cos n}$

Data $f(x) = 2\arctan x + \ln(1-2x) + 2x^2$, studiare, al variare di $a>0$, convergenza semplice e assoluta delle serie:

35) $\sum_{n=1}^{+\infty} (-1)^n f\left(\frac{1}{n^a}\right)$ 36) $\sum_{n=1}^{+\infty} f\left(\frac{(-1)^n}{n^a}\right)$

Data $f(x) = \sin(2x) + 2\ln(1-x) + x^2$, studiare, al variare di $a>0$, convergenza semplice e assoluta delle serie:

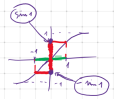
35) $\sum_{n=1}^{+\infty} (-1)^n f\left(\frac{1}{n^a}\right)$ 36) $\sum_{n=1}^{+\infty} f\left(\frac{(-1)^n}{n^a}\right)$

Per ogni $n\in\mathbb{N}-\{0\}$ definiamo $a_n = \begin{cases} \frac{2}{\sqrt[4]{n}} & \text{se } n \text{ è un quadrato perfetto} \\ -\frac{1}{\lfloor\sqrt{n}\rfloor^{\frac{5}{3}}} & \text{altrimenti} \end{cases}$ studiare convergenza semplice e assoluta di:

37) $\sum_{n=1}^{+\infty} a_n$ 38) $\sum_{n=1}^{+\infty} (a_n)^2$

39) $\sum_{n=1}^{+\infty} (a_n)^3$ 40) $\sum_{n=1}^{+\infty} \arctan(a_n)$

24 L.30 $\sum_{n=1}^{+\infty} \underbrace{(\sin(\sin n))^n}_{a_n}$



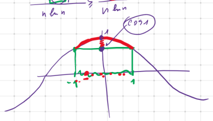
$\sqrt[n]{|a_n|} = |\sin(\sin n)| \le \boxed{\sin 1}$

$-1 \le \sin x \le 1$

SICCOME $\sqrt[n]{|a_n|} \le C < 1$ ALLORA $\sum |a_n|$ CONV. PER CR. RADICE, QUINDI $\sum a_n$ CONV. PER CRIT. CONV. ASS.

23 L.30 $\sum_{n=3}^{+\infty} \frac{\cos(\cos n)}{n\ln(n)}$

$\frac{\cos(\cos n)}{n\ln n} \ge \frac{c}{n\ln n}$ $(c>0)$ $(\cos 1)$



25 L.30 $\sum_{n=1}^{+\infty} \underbrace{\left(\arctan\frac{1}{\sqrt[4]{n}} - \sin\frac{1}{\sqrt[4]{n}}\right)}_{a_n}$ DIV. PER CONF. ASINT. CON

$a_n = \cancel{\frac{1}{\sqrt[4]{n}}} - \frac{1}{3}\frac{1}{\sqrt[4]{n^3}} + o\left(\frac{1}{\sqrt[4]{n^5}}\right) - \cancel{\frac{1}{\sqrt[4]{n}}} + \frac{1}{6}\cdot\frac{1}{\sqrt[4]{n^3}} + o\left(\frac{1}{\sqrt[4]{n^5}}\right) =$

$= \underbrace{-\frac{1}{6}\cdot\frac{1}{n^{\frac{3}{4}}}}_{b_n} + o\left(\frac{1}{\sqrt[4]{n^5}}\right)$

$\boxed{\frac{1}{2} < \frac{a_n}{b_n} < 2}$

$2b_n < a_n < \frac{1}{2}b_n$

29 L.30 $\sum_{n=2}^{+\infty} \frac{(n!)!}{(n^{n^n})}$

$a^{b^c} = a^{(b^c)}$

$a_1 \uparrow (a_2 \uparrow (a_3 \uparrow a_4))$

$(n^n)^{(n^n)} \gg (n^n)! \gg (n!)!$

$k^k \gg k!$ $2^{2^{2^{2}}}$

$\boxed{n^{n^n}} = n^{n\cdot n^{n-1}} = (n^n)^{n^{n-1}} = (n^{n-1}\cdot n)^{n^{n-1}} = (n^{n-1})^{n^{n-1}}\cdot n^{n^{n-1}} \ge$

$\ge (n!)^{n!}\cdot n^{n^{n-1}} \ge$

$\ge \boxed{(n!)!\cdot n^{n^{n-1}}}$

$n^n \gg n!$ n^{n-2}

$n^{n-1} \gg n!$

$\frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{n\cdot n\cdot n\cdots n} \le 1\cdot\frac{2}{n} \to 0$

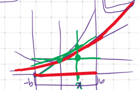
$0 < \frac{(n!)!}{n^{n^n}} \le \frac{\cancel{(n!)!}}{\cancel{(n!)!}\cdot n^{n^{n-1}}} = \boxed{\frac{1}{n^{n^{n-1}}}}$

DSS.1 $\sum_{n=0}^{+\infty} \frac{x^n}{n!}$ CONV. A e^x

$\forall m \in \mathbb{N}$ $\boxed{\sum_{n=0}^{m} \frac{x^n}{n!}}$ È $P_m(x)$ POL. DI TAYLOR DI e^x DI ORDINE m CENTRATO IN 0.

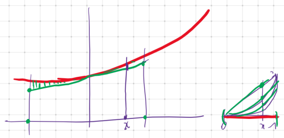
$\boxed{\lim_{m\to+\infty} P_m(x) = \boxed{e^x}}$ $\boxed{P_m(x)} = \left(1+x+\frac{x^2}{2}+\dots+\frac{x^m}{m!}\right)$ $\boxed{\sum \frac{x^m}{m!}} = e^x$

$\sum_{n=0}^{+\infty} \frac{3^n}{n!} = e^3$ $\sum_{n=0}^{+\infty} \frac{1}{n!} = e$



$\left|e^x - P_m(x)\right| = \left|\frac{f^{(m+1)}(\xi)}{(m+1)!}\cdot x^{m+1}\right| \le \boxed{\frac{e^b}{(m+1)!}\cdot b^{m+1}} \to 0$

$\sup_{x\in[-b,b]} \left|e^x - P_m(x)\right| \le \square$



32 L.30 $\sum_{n=1}^{+\infty} \underbrace{\boxed{e - \sum_{k=0}^{n} \frac{1}{k!}}}_{a_n}$

$1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} \dots = e$

$0 \le a_n \le \boxed{\frac{e}{(n+1)!}}$

$a_n = \sum_{k=n+1}^{+\infty} \frac{1}{k!} = \frac{1}{(n+1)!}\sum_{k=n+1}^{+\infty} \boxed{\frac{(n+1)!\,(k-(n+1))!}{k!}}\cdot\frac{1}{(k-(n+1))!} =$

$k \ge n+1$

$= \frac{1}{(n+1)!}\sum_{k=n+1}^{+\infty} \cancel{\frac{1}{\binom{k}{n+1}}}\;\frac{1}{(k-(n+1))!} \le$

$\le \frac{1}{(n+1)!}\cdot\sum_{k=n+1}^{+\infty} \frac{1}{(k-(n+1))!} =$

$= \frac{1}{(n+1)!}\boxed{\sum_{p=0}^{+\infty} \frac{1}{p!}} = \boxed{\frac{e}{(n+1)!}}$

46 .L.30 $\sum_{n=1}^{+\infty} \frac{1}{\sqrt[n]{(n!)^\alpha}}$

$\boxed{\alpha>1}$ $\boxed{\text{CONV.}}$

$\boxed{\alpha\le 1 \text{ DIV.}}$

$\boxed{n^n = o((n!)^\alpha)}$ $\boxed{\text{SE } \alpha>1}$

$\alpha \le 1$

$(n!)^\alpha \le n^n$

$\boxed{\frac{1}{\sqrt[n]{(n!)^\alpha}}} \ge \frac{1}{\sqrt[n]{n^n}} = \boxed{\frac{1}{n}}$

$\boxed{\delta>0}$ $A_n = \boxed{\frac{n^n}{(n!)^{1+\delta}}} \to 0$

$\sum A_n$ CONV.

$\frac{A_{n+1}}{A_n} = \frac{(n+1)^{n+1}}{((n+1)!)^{1+\delta}}\cdot\frac{(n!)^{1+\delta}}{n^n} = \frac{(n+1)^1}{(n+1)^{1+\delta}}\cdot\boxed{\frac{(n+1)^n}{n^n}} =$

$= \frac{1}{(n+1)^\delta}\cdot\left(1+\frac{1}{n}\right)^n \to 0$

$\alpha = 1+\delta$ $\boxed{\alpha>\beta>1}$ $\alpha>1$

$0 \le \boxed{\frac{1}{\sqrt[n]{((n!)^{\frac{\alpha}{\beta}})^\beta}}} = \left(\frac{1}{\sqrt[n]{(n!)^{\frac{\alpha}{\beta}}}}\right)^\beta \le \left(\frac{1}{\sqrt[n]{n^n}}\right)^\beta = (\text{CONV})$

$= \left(\frac{1}{n}\right)^\beta = \boxed{\frac{1}{n^\beta}}$