Analisi Matematica 2 - 19 Aprile 2022 - docente: Callegari

Lezione 20

Prova simulata di I Esonero

1 CALCOLARE $\int_{-\infty}^{+\infty} \ln\left(\frac{x^2+8x+20}{x^2+8x+17}\right) dx$

2 PER QUALI $\alpha>0$ CONVERGE $\int_0^{+\infty} \frac{\sin x^\alpha}{\arctan x^2}\, dx$?

3 PER QUALI $\alpha>0$ CONVERGE $\int_0^{+\infty} \frac{\sin x \cdot \sin\frac{1}{x+1}}{\ln^\alpha(1+x)}\, dx$?

4 STUDIARE AL VARIARE DI $\alpha\in\mathbb{R}$ LA CONVERGENZA DI $\sum_{n=1}^{+\infty} a_n$ E $\sum_{n=1}^{+\infty}(-1)^n a_n$

DOVE $a_n = \boxed{\frac{e^n\cdot(n-1)!}{n^{n-\alpha}}}$ $\qquad n! \approx \frac{n^n}{e^n}\sqrt{2\pi n}$ NON USARE

5 STUDIARE LA CONVERGENZA DI $\sum_{n=1}^{+\infty} a_n$ AL VARIARE DI $\alpha>0$ NEI SEGUENTI CASI:

$$a_n=\int_n^{n+1} f(x)\,dx \quad \text{E} \quad a_n=\int_n^{n+1} \cos\lfloor x\rfloor \cdot f(x)\,dx$$

DOVE:

$$f(x)=\frac{1+x^{2022}}{(1+x^\alpha)\cdot x^{2022}}$$

Consegnare entro la mezzanotte di domenica 24 aprile sul team del corso.

1

$$\int_{-\infty}^{+\infty} \ln\left(\boxed{\frac{x^2+8x+20}{x^2+8x+17}}\right) dx = \qquad \frac{x^2+8x+16+4}{x^2+8x+16+1} = \frac{(x+4)^2+4}{(x+4)^2+1}$$

$$= \dots = \int_{-\infty}^{+\infty} \ln\left(\frac{x^2+4}{x^2+1}\right) dx = 2\int_0^{+\infty} \ln\left(\frac{x^2+4}{x^2+1}\right) dx =$$

$$= 2\int_0^{+\infty} \ln(x^2+4)-\ln(x^2+1)\, dx =$$

$$= 2\lim_{c\to+\infty}\int_0^c \left(\ln(x^2+4)-\ln(x^2+1)\right)\cdot(x)'\, dx =$$

$$= 2\lim_{c\to+\infty}\left(\left[x\cdot\ln\frac{x^2+4}{x^2+1}\right]_0^c - \int_0^c\left(\frac{2x^2\overset{+8-8}{}}{x^2+4}-\frac{2x^2\overset{+2-2}{}}{x^2+1}\right)dx\right)=$$

$$= 2\lim_{c\to+\infty}\left(\left[x\cdot\ln\frac{x^2+4}{x^2+1}\right]_0^c - \int_0^c\left(\cancel{2}-\frac{8}{x^2+4}-\cancel{2}+\frac{2}{x^2+1}\right)dx\right)= \dots$$

$= \dots$

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2 $\int_0^{+\infty} \frac{\sin x^\alpha}{\arctan x^2}\, dx$ $\quad \boxed{\alpha>0}$ $\quad \boxed{\alpha>1}$

$$\to \text{①} = \int_0^1 \underbrace{\boxed{\frac{\sin x^\alpha}{\arctan x^2}}}_{f(x)} dx \qquad \text{②} = \int_1^{+\infty} \frac{\sin x^\alpha}{\arctan x^2}\, dx$$

PER $x\to 0^+$

$$f(x)\approx\frac{x^\alpha}{x^2}=\boxed{\frac{1}{x^{2-\alpha}}} \qquad \int \frac{1}{x^{2-\alpha}} \text{ CONV.} \iff 2-\alpha<1 \quad \boxed{\alpha>1}$$

$x=t^{\frac{1}{\alpha}}$

$$\text{②} = \int_1^{+\infty} \frac{\sin x^\alpha}{\arctan x^2}\, dx = \lim_{c\to+\infty}\int_1^c \frac{\sin x^\alpha}{\arctan x^2}\, dx =$$

$$= \lim_{c\to+\infty}\int_1^{c^\alpha} \frac{\sin t}{\arctan(t^{\frac{2}{\alpha}})}\cdot\frac{1}{\alpha}\cdot t^{\left(\frac{1}{\alpha}-1\right)}\, dt =$$

$$= \lim_{c\to+\infty}\frac{1}{\alpha}\int_1^{c^\alpha} \frac{\sin t}{\arctan(t^{\frac{2}{\alpha}})\cdot t^{1-\frac{1}{\alpha}}}\, dt =$$

$$= \frac{1}{\alpha}\cdot\int_1^{+\infty} \frac{\sin t}{\arctan(t^{\frac{2}{\alpha}})\, t^{1-\frac{1}{\alpha}}}\, dt$$

$g(t)=\arctan(t^{\frac{2}{\alpha}})\, t^{1-\frac{1}{\alpha}}$ $\quad \boxed{\text{CRESCE}}$

$g(t)\to+\infty$ PER $t\to+\infty$ (ovvio)

$$\to g'(t)=\left(\frac{1}{1+t^{\frac{4}{\alpha}}}\cdot t^{\frac{2}{\alpha}-1}\cdot\frac{2}{\alpha}\right)\cdot t^{1-\frac{1}{\alpha}} + \arctan(t^{\frac{2}{\alpha}})\cdot\left(1-\frac{1}{\alpha}\right)\cdot t^{-\frac{1}{\alpha}}$$

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5 $\boxed{\sum_{n=1}^{+\infty} a_n}$ $\quad \boxed{\alpha>1}$ $\quad (\alpha>0)$

$$a_n=\int_n^{n+1} \underbrace{\frac{1+x^{2022}}{(1+x^\alpha)\, x^{2022}}}_{f(x)}\, dx$$

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty}\int_1^{n+1} f(x)\,dx = \lim_{c\to+\infty}\int_1^c f(x)\,dx = \int_1^{+\infty} f(x)\,dx$$

$$S_n=(a_1+a_2+\dots+a_n) = \int_1^2 + \int_2^3 + \dots + \int_n^{n+1} = \int_1^{n+1} f(x)\,dx$$

$$\int_1^{+\infty} \underbrace{\frac{1+x^{2022}}{x^{2022}}\cdot\frac{1}{1+x^\alpha}}_{\approx\frac{1}{x^\alpha} \text{ PER } x\to+\infty}\, dx$$

$$\int_1^{+\infty}\frac{1}{x^\alpha}\,dx \text{ CONV.} \iff \alpha>1$$

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$$\sum_{n=1}^{+\infty} b_n \qquad b_n=\int_n^{n+1}\cos\overbrace{\lfloor x\rfloor}^{n} f(x)\,dx = \cos n\cdot\int_n^{n+1} f(x)\,dx = \cos n\cdot a_n$$

$$\sum_{n=1}^{+\infty} (\cos n)\cdot\boxed{a_n}$$

$B_n=\cos 1+\cos 2+\dots+\cos n$ $\quad (B_n)$ è limitata (NOTO)

$\boxed{\alpha>0}$

$$\boxed{a_n}=\int_n^{n+1}\frac{1+x^{2022}}{(1+x^\alpha)\, x^{2022}}\, dx =$$

$$= \int_n^{n+1}\left(1+\frac{1}{x^{2022}}\right)\cdot\frac{1}{1+x^\alpha}\, dx \le 2\int_n^{n+1}\frac{1}{1+n^\alpha}\, dx =$$

$$= \frac{2}{1+n^\alpha}\cdot\int_n^{n+1} 1\, dx = \boxed{\frac{2}{1+n^\alpha}}$$

$$0\le a_n\le\frac{2}{1+n^\alpha}$$

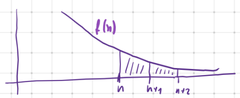
$\downarrow \quad \downarrow \quad \downarrow$

$0 \quad 0 \quad 0$

$\boxed{a_{n+1}\le a_n}$??

$\int_{n+1}^{n+2} f(x)\,dx \qquad \int_n^{n+1} f(x)\,dx$

$$f(x)=\left(1+\frac{1}{x^{2022}}\right)\cdot\frac{1}{1+x^\alpha}$$



$$a_{n+1}=\int_{n+1}^{n+2} f(x)\,dx = \int_n^{n+1} f(x+1)\,dx$$

$$a_n=\int_n^{n+1} f(x)\,dx$$

$f(x)\ge f(x+1) \quad \forall x\in[n,n+1]$

$\int \ \ge \int \dots$

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3 $\sum_{n=2}^{+\infty}\underbrace{\boxed{\frac{e^n\cdot(n-1)!}{n^{n-\alpha}}}}_{a_n}$

$$\boxed{\frac{a_{n+1}}{a_n}} = \frac{e^{\cancel{n+1}}\cdot n\cancel{!}}{(n+1)^{n+1-\alpha}}\cdot\frac{n^{n-\alpha}}{\cancel{e^n}\cdot\cancel{(n-1)!}} = e\cdot\frac{n^{n+1}}{(n+1)^{n+1}}\cdot\left(\frac{n}{n+1}\right)^{-\alpha} =$$

$$= e\cdot\left(\frac{n+1}{n}\right)^\alpha\frac{1}{\left(\frac{n+1}{n}\right)^{n+1}} =$$

$$= \left(1+\frac{1}{n}\right)^\alpha\cdot\frac{e}{\left(1+\frac{1}{n}\right)^{n+1}} =$$

$$= e^{\alpha\ln\left(1+\frac{1}{n}\right)}\cdot e^1\cdot e^{-(n+1)\ln\left(1+\frac{1}{n}\right)} =$$

$$= e^{\boxed{\alpha\ln\left(1+\frac{1}{n}\right)+1-(n+1)\ln\left(1+\frac{1}{n}\right)}} = e^{\left(\alpha-\frac{1}{2}\right)\cdot\frac{1}{n}+o\left(\frac{1}{n^2}\right)} =$$

$$(*) = \alpha\left(\frac{1}{n}-\frac{1}{2n^2}+o\left(\frac{1}{n^3}\right)\right)+1+(n+1)\left(-\frac{1}{n}+\frac{1}{2n^2}+o\left(\frac{1}{n^3}\right)\right)=$$

$$= \frac{\alpha}{n}-\frac{\alpha}{2}\cdot\frac{1}{n^2}+o\left(\frac{1}{n^3}\right)+\cancel{1}\cancel{-1}+\frac{1}{2}\cdot\frac{1}{n}+o\left(\frac{1}{n^2}\right)-\frac{1}{n}+o\left(\frac{1}{n^2}\right)+o\left(\frac{1}{n^3}\right)=$$

$$= \left(\alpha-\frac{1}{2}\right)\cdot\frac{1}{n}+o\left(\frac{1}{n^2}\right)$$

$$= 1+\left(\alpha-\frac{1}{2}\right)\cdot\frac{1}{n}+o\left(\frac{1}{n^2}\right)+o\left(\square^2\right) = \qquad o\left(\frac{1}{n^2}\right)$$

$$= \boxed{1-\frac{\frac{1}{2}-\alpha}{n}+o\left(\frac{1}{n^2}\right)}$$

$$\frac{a_{n+1}}{a_n} = 1-\frac{\left(\frac{1}{2}-\alpha\right)}{2}+o\left(\frac{1}{n}\right)$$

$\frac{1}{2}-1>\alpha \quad \boxed{\alpha<-\frac{1}{2}}$

$\left(\frac{1}{2}-\alpha\right)>1 \quad \sum$ CONV.

$\left(\frac{1}{2}-\alpha\right)<1 \quad \sum$ DIV.

$\boxed{\alpha>-\frac{1}{2}}$