Analisi Matematica 2 - 22 Aprile 2022 - docente: Callegari

Lezione 21

Esercizi sulle Serie

STUDIARE IL CARATTERE DI $\sum a_n$ E $\sum (-1)^n a_n$ (EVENTUALMENTE AL VARIARE DI α) NEI SEGUENTI CASI.

1) $a_n = \int_0^{\pi} \sin((2n+1)x)\,dx$

2) $a_n = \int_0^1 (\sin x)^n\,dx$

3) $a_n = \int_0^{\alpha} (\sin x)^n\,dx \quad \left(0<\alpha<\frac{\pi}{2}\right)$

4) $a_n = \int_0^{\frac{\pi}{2}-\frac{1}{n}} (\sin x)^n\,dx$

5) $a_n = \int_0^{\frac{\pi}{2}} (\sin x)^n\,dx$

6) $a_n = \int_0^1 \sin(x^n)\,dx$

7) $a_n = \int_n^{+\infty} e^{-x}\,dx$

8) $a_n = \int_0^{+\infty} e^{-nx}\,dx$

9) $a_n = \int_n^{+\infty} e^{-x^2}\,dx$

10) $a_n = \int_n^{+\infty} \frac{1}{x^{\alpha}}\,dx \quad (\alpha>0)$

11) $a_n = \int_n^{+\infty} \frac{1}{\sin x + x\sqrt{x}}\,dx$

12) $a_n = \int_{n+\sin n}^{+\infty} \frac{1}{\sqrt[5]{1+x^6}}\,dx$

13) $a_n = \int_{n+10\sin n}^{+\infty} \frac{1}{\sqrt[3]{x^6+x^3}}\,dx$

STUDIARE IL CARATTERE DELLE SEGUENTI SERIE:

14) $\sum_{n=1}^{+\infty} \frac{(n!)^{\alpha}}{n^{4n}} \quad (\alpha>0)$

15) $\sum_{n=1}^{+\infty} \frac{n!}{n^n}\cdot\left(\alpha+\frac{1}{n}\right)^n \quad (\alpha>0)$

16) $\sum_{n=1}^{+\infty} (-1)^n \frac{\sin\sqrt{n}}{n^{\alpha}} \quad (\alpha>0)$

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1) $a_n = \int_0^{\pi} \sin((2n+1)x)\,dx = \quad \boxed{\sum a_n} \quad \sum(-1)^n a_n$

$= \left[\frac{-\cos((2n+1)x)}{2n+1}\right]_0^{\pi} =$

$= \frac{1}{2n+1}(1+1) = \frac{2}{2n+1}$

$\sum \frac{2}{2n+1}$ DIV.

$\sum (-1)^n \cdot \frac{2}{2n+1}$ CONV. LEIB.

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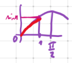
2) $a_n = \int_0^1 (\sin x)^n\,dx \qquad \sum a_n$



$0 \le a_n = \int_0^1 (\sin x)^n\,dx \le \int_0^1 (\sin 1)^n\,dx =$

$= (\sin 1)^n \cdot \int_0^1 1\,dx = (\sin 1)^n$

$0 \le a_n \le (\sin 1)^n$, $q = \sin 1$

$\sum q^n$ CONV. PERCHÉ $q = \sin 1 < 1$

$\sum(-1)^n a_n$ CONV. PER IL CR. CONV. ASS. POICHÉ $\sum |(-1)^n a_n| = \boxed{\sum a_n}$

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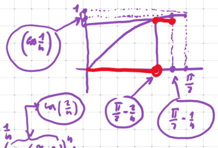
4) $\sum_{n=1}^{+\infty} a_n \qquad a_n = \int_0^{\frac{\pi}{2}-\frac{1}{n}} (\sin x)^n\,dx$



$a_n \ge \int_{\frac{\pi}{2}-\frac{2}{n}}^{\frac{\pi}{2}-\frac{1}{n}} \left(\sin\left(\frac{\pi}{2}-\frac{2}{n}\right)\right)^n dx \ge$

$\ge \left(\cos\left(\frac{2}{n}\right)\right)^n \cdot \frac{1}{n} \approx \frac{1}{n}$, $\left(\cos\left(\frac{2}{n}\right)\right)^n \to 1$

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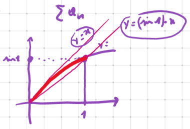
6) $a_n = \int_0^1 \sin(x^n)\,dx \qquad \sum a_n$

$\boxed{\sum(-1)^n a_n}$



$(\sin 1)\cdot x^n \le \sin(x^n) \le x^n$

$\int_0^1 (\sin 1)\cdot x^n\,dx \le a_n \le \int_0^1 x^n\,dx$

$\int_0^1 x^n\,dx = \left[\frac{1}{n+1}\cdot x^{n+1}\right]_0^1 = \frac{1}{n+1}$

$\sum$ DIV.

$(\sin 1)\cdot\int_0^1 x^n\,dx \le a_n \le \int_0^1 x^n\,dx$

$(\sin 1)\cdot\frac{1}{n+1} \le a_n \le \frac{1}{n+1} \Rightarrow \sum a_n$ DIV. PER CONF.

$a_{n+1} = \int_0^1 \boxed{\sin(x^{n+1})}\,dx$

$a_n = \int_0^1 \boxed{\sin(x^n)}\,dx$

$x \in [0,1]$

$0 \le x^{n+1} \le x^n \le 1$?

$\sin x^{n+1} \le \sin x^n$

$\int_0^1 \sin(x^{n+1})\,dx \le \int_0^1 \sin(x^n)\,dx$

$a_{n+1} \le a_n$

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7) $\sum a_n$ $\boxed{\text{CONV.}}$

$a_n = \int_n^{+\infty} e^{-x}\,dx = \dots =$

$= \left[-e^{-x}\right]_n^{+\infty} = 0 + e^{-n} = \left(\frac{1}{e}\right)^n$

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9) $\sum a_n \qquad a_n = \int_n^{+\infty} e^{-x^2}\,dx \le \left(\frac{1}{e}\right)^n$

CONV. PER CONF.

$\boxed{e^{-x^2} \le e^{-x}}$ $\quad \boxed{x \ge 1}$

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10) $\sum a_n \qquad a_n = \int_n^{+\infty} \frac{1}{x^{\alpha}}\,dx \quad (\alpha>0)$, >1

$x^{-\alpha}$

$\boxed{a_n} = \left[\frac{1}{1-\alpha}\, x^{-\alpha+1}\right]_n^{+\infty} =$

$= -\frac{1}{1-\alpha}\cdot n^{-\alpha+1} =$

$= \frac{1}{\alpha-1}\cdot\frac{1}{n^{\alpha-1}}$

$\sum a_n = \sum \circ = \frac{1}{\alpha-1}\sum\frac{1}{n^{\alpha-1}}$

$\alpha-1>1$

$\alpha>2$

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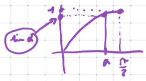
3) $\sum a_n \quad \sum(-1)^n a_n \qquad 0<\alpha<\frac{\pi}{2}$

$a_n = \int_0^{\alpha} (\sin x)^n\,dx$



$0 \le a_n = \int_0^{\alpha} (\sin x)^n\,dx \le \int_0^{\alpha} (\sin\alpha)^n\,dx =$

$= (\sin\alpha)^n \cdot \int_0^{\alpha} 1\,dx = \alpha(\sin\alpha)^n$

$0 \le a_n \le \boxed{\alpha\cdot q^n} \qquad \boxed{q = \sin\alpha < 1}$

$\sum \alpha q^n = \alpha\sum q^n$ CONV.

$\sum a_n$ CONV. PER. CR. CONF.

$\sum(-1)^n a_n$ CONV. PER. C. ASS. CONV.

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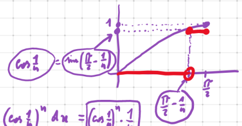
5) $\sum a_n \qquad a_n = \int_0^{\frac{\pi}{2}} (\sin x)^n\,dx \qquad \boxed{\sum(-1)^n a_n}$



$\boxed{a_n} \ge \int_{\frac{\pi}{2}-\frac{1}{n}}^{\frac{\pi}{2}} \left(\cos\frac{1}{n}\right)^n dx = \boxed{\left(\cos\frac{1}{n}\right)^n\cdot\frac{1}{n}}$

$\left(\cos\frac{1}{n}\right)^n = \left(1-\left(1-\cos\frac{1}{n}\right)\right)^n$

$\left(\left(1-\frac{1}{n^2}\right)^{n^2}\right)^{\frac{1}{n}} \to 1$, $\left(1-\frac{1}{n^2}\right)^{n^2} \to \frac{1}{e}$, $\frac{1}{n} \to 0$

$\sum\left(\cos\frac{1}{n}\right)^n\cdot\frac{1}{n}$

STESSO CAR. (?) — $\sum\frac{1}{n}$ DIV

$\left(\cos\frac{1}{n}\right)^n = e^{n\ln\left(\cos\frac{1}{n}\right)} =$

$= e^{n\ln\left(1+\left(\cos\frac{1}{n}-1\right)\right)}$

$\boxed{\cos\frac{1}{n}-1} = \cancel{1}-\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right)\cancel{-1}$

$= -\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right)$

$= e^{n\cdot\left(\boxed{\cos\frac{1}{n}-1}+o\left(\boxed{\cos\frac{1}{n}-1}\right)\right)} =$

$= e^{n\cdot\left(-\frac{1}{2n^2}+o\left(\frac{1}{2n^2}\right)\right)} = e^{-\frac{1}{2n}+o\left(\frac{1}{n}\right)} \to 1$, $-\frac{1}{2n}+o\left(\frac{1}{n}\right) \to 0$

$\sin(x^n) \qquad \sin(x^{n+1})$

$0 \le x^{n+1} \le x^n \le 0$

$\sin(x^{n+1}) \le \sin(x^n)$

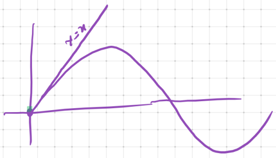
$\int_0^1 \circ \le \int_0^1 \circ$

$a_{n+1} \le a_n$

$(\sin x)'$

$\cos x$

$\cos(0)=1$



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14) $\sum \boxed{\frac{(n!)^{\alpha}}{n^{4n}}}$, a_n

$\frac{a_{n+1}}{a_n} = \frac{((n+1)!)^{\alpha}}{(n+1)^{4n+4}}\cdot\frac{n^{4n}}{(n!)^{\alpha}} =$

$= \boxed{\frac{(n+1)^{\alpha}}{(n+1)^4}}\cdot\boxed{\frac{n^{4n}}{(n+1)^{4n}}} =$

$= \boxed{}\cdot\frac{1}{\left(\left(1+\frac{1}{n}\right)^n\right)^4}$, $\frac{1}{\left(\left(1+\frac{1}{n}\right)^n\right)^4} \to \frac{1}{e^4}$

$\to +\infty \quad \alpha>4$

$\to \frac{1}{e^4} \quad \alpha=4$

$\to 0 \quad \alpha<4$