EXE 12 - NUMERI COMPLESSI (3 - MAGGIO 2023)

1. MOSTRARE CHE $\lim_{n \to +\infty} \left(1+\frac{z}{n}\right)^n = e^z \quad (z \in \mathbb{C})$

2. **3.** Siano dati i due numeri complessi $z_1 = -\sqrt{3} + i$ e $z_2 = \sqrt{2} + i\sqrt{2}$ e sia $z \in \mathbf{C}$ tale che esista n intero positivo tale che sia z_1 che z_2 sono radice ennesima di z.

   (a) Calcolare z_1^{96} e z_2^{96};

   (b) Qual è il minimo valore che può avere $|z|$?

3. **4.** Dati i due numeri complessi $z_1 = \sqrt{2+\sqrt{3}} + i\sqrt{2-\sqrt{3}}$ e $z_2 = \sqrt{2-\sqrt{2}} - i\sqrt{2+\sqrt{2}}$

   (a) calcolare z_1^{2022} e, in particolare, trovare $\left\lfloor \Im\left(z_1^{2022}\right) \right\rfloor$, ovvero la parte intera della parte immaginaria di z_1^{2022};

   (b) determinare l'insieme $\mathcal{I} = \left\{ n \in \mathbf{Z} \mid z_1^n = z_2^n \right\}$.

4. **4.** Dopo aver risolto in $\mathbf{C}$ l'equazione $z^{18} = z^{10}$, trovare tutti in numeri complessi per i quali l'insieme delle radici decime e l'insieme delle radici diciottesime non sono disgiunti.

5. MOSTRARE CHE $z_1, z_2, z_3 \in \mathbb{C}$ SONO VERTICI DI UN TRIANGOLO EQUILATERO SE E SOLO SE $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$

6. DIMOSTRARE IL T. DI NAPOLEONE [...] USANDO I NUMERI COMPLESSI

7. DESCRIVERE I SEGUENTI INSIEMI

$$A = \{ z \in \mathbb{C} \mid |z+i| + |z-i| = 4 \}$$

$$A = \left\{ z \in \mathbb{C} \mid \arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4} \right\}$$

$$A = \{ z \in \mathbb{C} \mid |z-1| = 2|z| \}$$

(1)

$$\boxed{\lim_{n\to+\infty}\left(1+\frac{z}{n}\right)^n = e^z}$$

$z = a+ib$

$$\lim_{n\to+\infty}\boxed{\left(1+\frac{a+ib}{n}\right)^n} = \qquad e^a(\cos b + i\sin b) = \boxed{e^{a+ib}}$$

$$\boxed{e^a}$$

$$\boxed{b}$$

$$|\,| = \left|\left(1+\frac{a}{n}+i\frac{b}{n}\right)^n\right| =$$

$$= \left|1+\frac{a}{n}+i\frac{b}{n}\right|^n =$$

$$= \left(\sqrt{\left(1+\frac{a}{n}\right)^2+\left(\frac{b}{n}\right)^2}\right)^n =$$

$$= \left(1+\frac{a^2}{n^2}+\underbrace{2\frac{a}{n}}+\frac{b^2}{n^2}\right)^{\frac{n}{2}} =$$

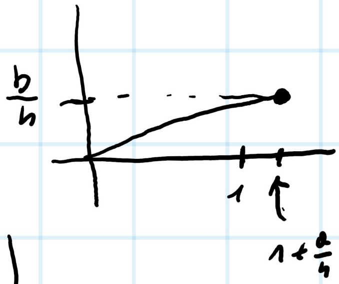
$$= \left(1+\frac{a+\boxed{\frac{a^2}{2n}+\frac{b^2}{2n}}}{\frac{n}{2}}\right)^{\frac{n}{2}} =$$

$$= \left(1+\frac{a}{\frac{n}{2}}\cdot\left(1+\frac{a}{2n}+\frac{b}{2n}\right)\right)^{\frac{n}{2}}$$

$$= \left(1+\frac{a}{\frac{n}{2}\cdot\frac{1}{1+\frac{a}{2n}+\frac{b}{2n}}}\right)^{\frac{n}{2}} \to e^a$$

$$\boxed{\left(1+\frac{a+bi}{n}\right)^n}$$

$$\underbrace{1+\frac{a+bi}{n}}_{} = \underbrace{\underbrace{1+\frac{a}{n}}_{}+i\frac{b}{n}}_{}$$

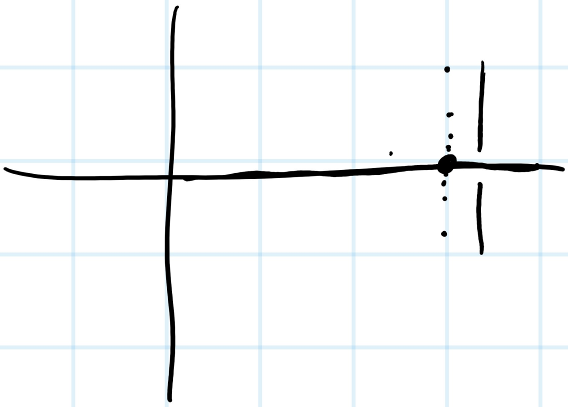


$$\arg(\ ) = \arctan\left(\frac{\frac{b}{n}}{1+\frac{a}{n}}\right)$$

$$\arg(\ ) = \underbrace{n \cdot \arctan\left(\frac{\frac{b}{n}}{1+\frac{a}{n}}\right)}_{}$$

$$\lim_{n\to+\infty} n \arctan\left(\frac{\frac{b}{n}}{1+\frac{a}{n}}\right) = \lim_{n\to+\infty} n \cdot \frac{\frac{b}{n}}{1+\frac{a}{n}} = b$$

$$\boxed{\cos\theta = \frac{a}{\sqrt{a^2+b^2}} \quad \sin\theta = \frac{b}{\sqrt{a^2+b^2}}} \qquad \theta \in [0, 2\pi)$$



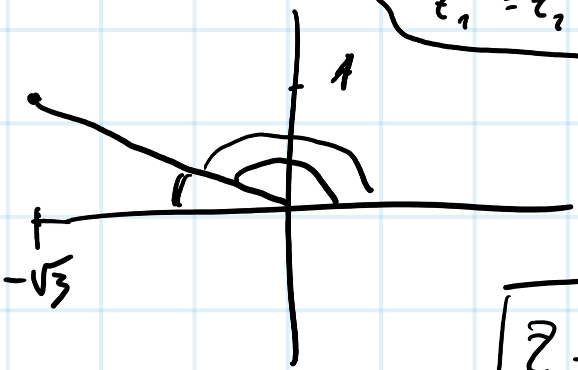
3. Siano dati i due numeri complessi $z_1 = -\sqrt{3}+i$ e $z_2 = \sqrt{2}+i\sqrt{2}$ e sia $z \in \mathbf{C}$ tale che esista n intero positivo tale che sia z_1 che z_2 sono radice ennesima di z.

(**a**) Calcolare z_1^{96} e z_2^{96};

(**b**) Qual è il minimo valore che può avere $|z|$?

QUAL È IL MINIMO $n \in \mathbb{N}-\{0\}$ t.c. $z_1^n = z_2^n$

$z_1 =$



$|z_1| = 2$

$\arg(z_1) = \frac{5}{6}\pi$

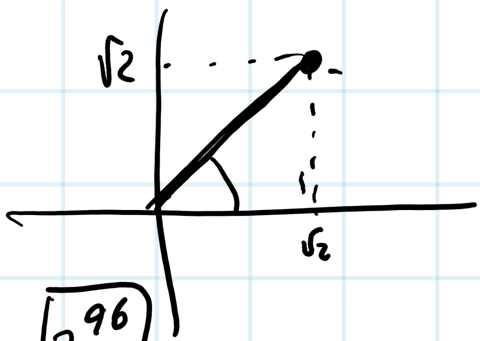
$$\boxed{z = 2^{96}}$$

$$z_1 = 2e^{i\frac{5}{6}\pi}$$

$5 \cdot 16$

$$z_1^{96} = \left(2e^{i\frac{5}{6}\pi}\right)^{96} = 2^{96} \cdot e^{80\pi i} = \boxed{2^{96}}$$

$$z_2 = \sqrt{2} + i\sqrt{2}$$



$$z_2 = 2e^{\frac{\pi}{4}i}$$

$$z_2^{96} = 2^{96} \cdot e^{96\frac{\pi}{4}i} = \boxed{2^{96}}$$

$$z_1^n = z_2^n$$

$$\left(\cancel{2}e^{\frac{5}{6}\pi i}\right)^n = \left(\cancel{2}e^{\frac{\pi}{4}i}\right)^n$$

$$\left(n \cdot \frac{5}{6}\pi - n \cdot \frac{\pi}{4}\right) = 2k\pi$$

$$n\left(\frac{5}{6} - \frac{1}{4}\right)\pi = 2k\pi$$

$$n\,\frac{7}{12}\pi = 2k\pi$$

$$\boxed{n = 24}$$

$$\boxed{2^{24}}$$

4. Dati i due numeri complessi $\quad z_1 = \sqrt{2+\sqrt{3}} + i\sqrt{2-\sqrt{3}} \quad$ e $\quad z_2 = \sqrt{2-\sqrt{2}} - i\sqrt{2+\sqrt{2}}$

(a) calcolare z_1^{2022} e, in particolare, trovare $\lfloor \Im\left(z_1^{2022}\right) \rfloor$, ovvero la parte intera della parte immaginaria di z_1^{2022};

(b) determinare l'insieme $\mathcal{I} = \{n \in \mathbf{Z} \mid z_1^n = z_2^n\}$.

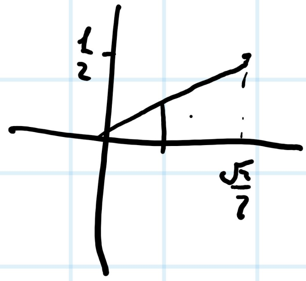
$$z_1^2 = \left(\sqrt{2+\sqrt{3}} + i\sqrt{2-\sqrt{3}}\right)^2 =$$

$$= \cancel{2} + \sqrt{3} - (\cancel{2} - \sqrt{3}) + 2i\sqrt{\underbrace{(2+\sqrt{3})(2-\sqrt{3})}_{4-3}} =$$

$$= 2\sqrt{3} + 2i =$$

$$= 4\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) =$$

$$= \boxed{4 \cdot \left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right)} \quad z_1^2$$



$$z_1 = \boxed{2\left(\cos\frac{\pi}{12} + i \sin\frac{\pi}{12}\right)} = \boxed{2e^{\frac{\pi}{12}i}}$$

$$\boxed{2\left(\cos\frac{13\pi}{12} + i \sin\frac{13}{12}\pi\right)}$$

$$2022 \cdot \frac{\pi}{12} = 2k\pi + \frac{\pi}{3}$$

$$\boxed{z_1^{2022}} = \boxed{2^{2022}} \cdot e^{\left(\frac{\pi}{12} \cdot 2022\, i\right)} = e^{\boxed{\frac{336\pi}{2}} i + \frac{\pi}{2} i}$$

$$= e^{\frac{\pi}{2} i} = i$$

1800

280

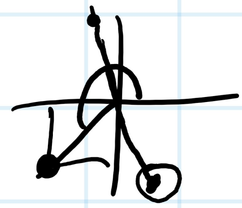
40

336 + 1

$$\boxed{z_1^{2022} = 2^{2022} \cdot i}$$

$$z_2 = \underbrace{\sqrt{2-\sqrt{2}}} - i\underbrace{\sqrt{2+\sqrt{2}}}$$

$$z_2^2 = \left(\underbrace{\sqrt{2-\sqrt{2}} - i\sqrt{2+\sqrt{2}}}\right)^2 =$$



$$= \cancel{2} - \sqrt{2} - (\cancel{2}+\sqrt{2}) - 2i\underbrace{\sqrt{(2-\sqrt{2})(2+\sqrt{2})}}$$

$$= \underbrace{-2\sqrt{2} - i2\sqrt{2}} = 4\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = \boxed{\underbrace{4\left(\cos\frac{5}{4}\pi + i\sin\frac{5}{4}\pi\right)}}$$

$$z_2^2 = 4 \cdot$$

$$\to z_1 = 2\left(\cos\frac{5}{8}\pi + i\sin\frac{5}{8}\pi\right) = \boxed{2e^{\frac{5}{8}\pi i}}$$

$$\to z_2 = \underbrace{2\left(\cos\frac{13}{8}\pi + i\sin\frac{13}{8}\pi\right)} = \boxed{2e^{\frac{13}{8}\pi i}}$$

$$z_2 = 2e^{\frac{13}{8}\pi i}$$

$$z_1^n = z_2^n$$

$$\underbrace{\left(\cancel{2}e^{\frac{\pi}{12}i}\right)^n} = \underbrace{\left(\cancel{2}e^{\frac{13}{8}\pi i}\right)^n}$$

$$e^{\boxed{\frac{n}{12}\pi i}} = e^{\boxed{n\frac{13}{8}\pi i}}$$

$$n\left(\frac{\pi}{12} - \frac{13}{8}\pi\right) = 2k\pi$$

$$\|$$

$$n\left(\frac{2-39}{24}\right)\pi$$

$$-\boxed{n}\,\frac{37}{24}\,\pi$$

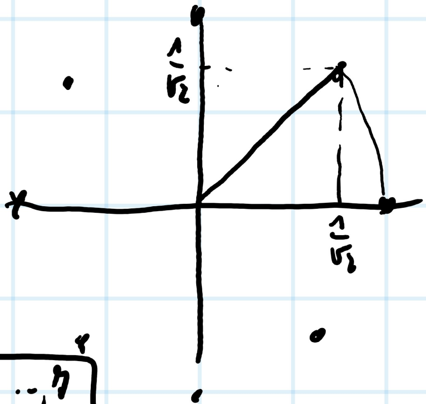
$$\boxed{n = 48k \qquad k \in \mathbb{Z}}$$

4. Dopo aver risolto in **C** l'equazione $z^{18} = z^{10}$, trovare tutti in numeri complessi per i quali l'insieme delle radici decime e l'insieme delle radici diciottesime non sono disgiunti.

$$z^{10} \cdot (z^8 - 1) = 0$$

$$\boxed{z = 0} \qquad z^8 - 1 = 0$$

$$z^8 = 1$$



$$z_1 = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$$

$$A = \left\{\cos\left(k\frac{\pi}{4}\right) + i\sin\left(k\frac{\pi}{4}\right) \;\middle|\; k = 0, \dots, 7\right\}$$

$$A = \left\{1, e^{\frac{\pi}{4}i}, e^{\frac{\pi}{2}i}, \dots, e^{\frac{7}{8}\pi i}\right\}$$

$$B = \{ z \in \mathbb{C} \mid \text{BUONI} \} =$$

$$= \left\{ 1^{10}, \left(e^{\frac{\pi}{4} i}\right)^{10}, \dots \left(e^{\frac{7}{4}\pi i}\right)^{10} \right\}$$

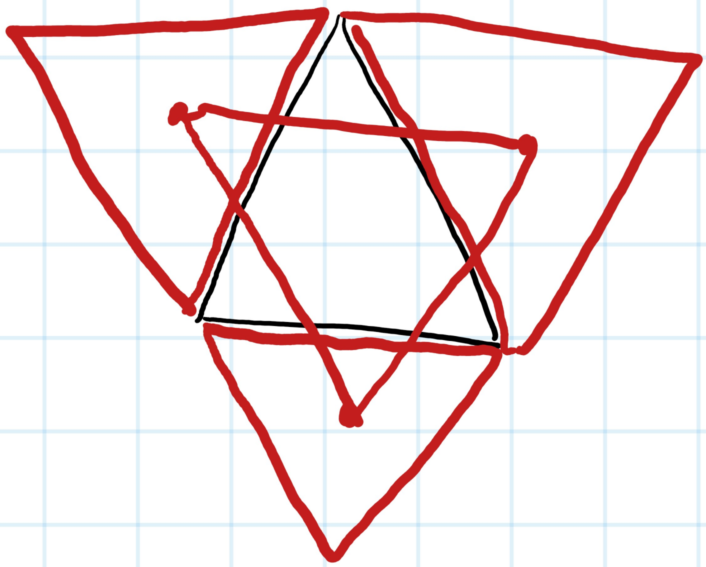
$$\left(e^{\frac{\pi}{4} i + 2k \frac{\pi}{4} i}\right)^{10} = \qquad k = 0, \dots, 7$$

$$= e^{\frac{5}{2}\pi i + \boxed{k \frac{5}{2}\pi} i}$$

$$\underbrace{e^{\frac{5}{2}\pi i} \quad e^{5\pi i} \quad e^{\frac{15}{2}\pi i} \quad e^{10\pi i}}$$

$$\underbrace{0 \quad \underbrace{e^{\frac{\pi}{2} i} \quad e^{\pi i} \quad e^{\frac{3}{2}\pi i} \quad 1}}$$

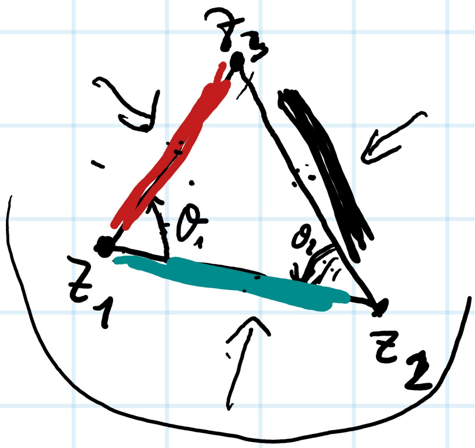
$$\{0, i, -1, -i, 1\}$$



OSS.

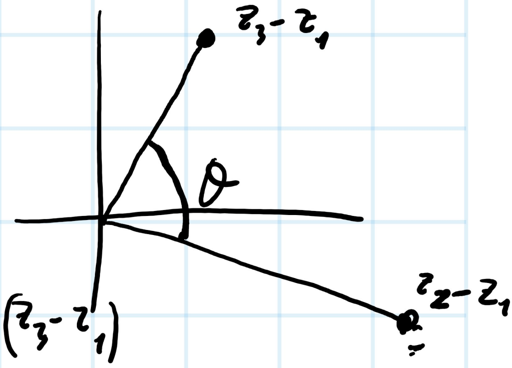
DATI $z_1, z_2, z_3 \in \mathbb{C}$ SONO VERTICI DI UN TR. EQUI

$$\Longleftrightarrow \quad z_1^2+z_2^2+z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$



DIM

$$\theta_1 = \arg\left(\overbrace{\frac{z_3 - z_1}{z_2 - z_1}}^{w}\right)$$



$$w(z_2 - z_1) = (z_3 - z_1)$$

$$\theta_2 = \arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right)$$

$$(*) \quad \boxed{\underbrace{\frac{z_3 - z_1}{z_2 - z_1}}_{w} = \underbrace{\frac{z_1 - z_2}{z_3 - z_2}}_{w}}$$

$$\frac{|z_3 - z_1|}{|z_2 - z_1|} = |w|$$

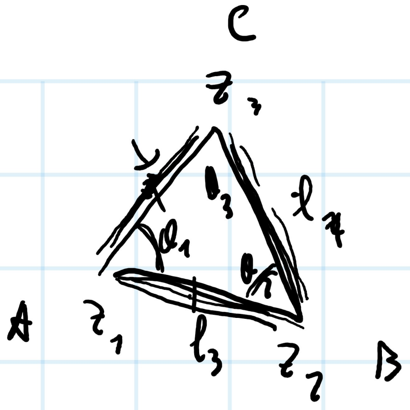
$$\boxed{|z_1 - z_2| = |w| \cdot |z_3 - z_2|}$$

$$\boxed{|z_3 - z_1| = |w| \cdot |z_2 - z_1|}$$

$$(*) \quad (z_3 - z_1)(z_3 - z_2) = \overbrace{(z_2 - z_1)(z_1 - z_2)}^{-(z_1 - z_2)^2}$$

$$z_3^2 - z_1 z_3 - z_2 z_3 + z_1 z_2 = -z_1^2 - z_2^2 + 2 z_1 z_2$$

$$\boxed{z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 \cdot z_1}$$



ABC è equilatero

$\Updownarrow$

$\theta_1 = \theta_2 \quad l_3 = l_1$

$\Updownarrow$

$\arg(w_1) = \arg(w_2)$

$|w_1| = |w_2|$

$\Longleftrightarrow$

$$\underbrace{\frac{z_3 - z_1}{z_2 - z_1}}_{w_1} = \underbrace{\frac{z_1 - z_2}{z_3 - z_2}}_{w_2}$$