# AM2 - EXE (10-05-2023) LIMITI IN 2 VARIABILI

**1** PER QUALI $\alpha, \beta \geq 0$ VALE $|x|^{\alpha} \cdot |y|^{\beta} = o(x^4 + y^4)$

CALCOLARE I SEGUENTI LIMITI:

**2** $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^{100} \cdot y^{100}}{x^{102} + y^{102} + (xy)^{60}}$

**3** $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^{100} y^{100}}{x^3 + y^3}$

**4** $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x y^5}{x^2 + y^8 + \underline{x y^6}}$

**5** $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^{\alpha} y^3}{(x^{30} + y^{18})(x^{10} + y^{10})}$ $(\alpha \geq 0)$

**6** $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^{100} y^{100}}{x^3 + y^3 + \underline{x^6 + y^6}}$

**7** $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2 + y^6}{x^4 + y^8 - (xy)^3}$

**8** $\displaystyle \lim_{(x,y)\to\infty} \frac{1}{x^2 + y^2 + \alpha \cdot xy}$ $(\alpha \in \mathbb{R})$

**9** $\displaystyle \lim_{(x,y)\to\infty} \frac{xy}{2 + x^2 y^2}$

**10** $\displaystyle \lim_{(x,y)\to\infty} \frac{xy^2}{x^4 + y^4 + xy}$

**DEF.** DATE $f, g : \Omega \to \mathbb{R}$ ($\Omega \subset \mathbb{R}^n$) $\bar{x}$ DI ACC. PER $\Omega$.

INOLTRE SIANO $f, g$ INFINITESIME PER $x \to \bar{x}$

DIREMO CHE $f(x) = o(g(x))$ SE

→ $$\lim_{x \to \bar{x}} \frac{f(x)}{g(x)} = 0$$

→ $f(x) = g(x) \cdot h(x)$

DOVE $h : \Omega \to \mathbb{R}$ È INFINITESIMA PER $x \to \bar{x}$

**1**

PER $(x, y) \to (0,0)$

$$|x|^\alpha \cdot |y|^\beta = o(x^4 + y^4) \quad ?$$

$\alpha, \beta \geq 0$

$$\lim_{(x,y) \to (0,0)} \frac{|x|^\alpha \cdot |y|^\beta}{x^4 + y^4} = l$$

| | |
|---|---|
| $\alpha + \beta > 4$ | $l = 0$ |
| $\alpha + \beta \leq 4$ | $l$ N.E. |

$\alpha + \beta = 4$

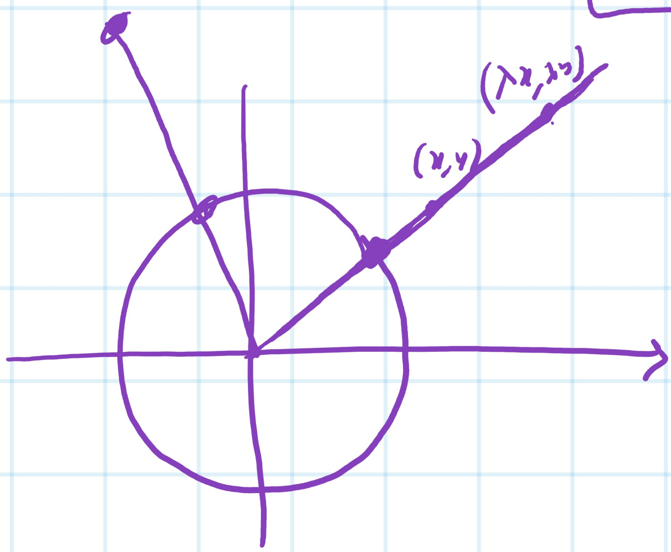
$$g(x,y) = \frac{|x|^\alpha \cdot |y|^\beta}{x^4 + y^4}$$

**DEF.** $g : \mathbb{R}^n - \{\bar{0}\} \to \mathbb{R}$ DICO CHE $g$ È POS. OMOG. DI GRADO $\alpha$ SE $\forall x \in \mathbb{R}^n - \{\bar{0}\}$ E $\forall \lambda > 0$

$$g(\lambda x) = \lambda^\alpha g(x)$$

---

$$g(\lambda x, \lambda y) = \frac{|\lambda x|^\alpha \cdot |\lambda y|^\beta}{(\lambda x)^4 + (\lambda y)^4} = \lambda^{\alpha+\beta-4} \underbrace{\frac{|x|^\alpha \cdot |y|^\beta}{x^4 + y^4}}_{g(x,y)} =$$



$0 < a \le \alpha$

$0 < b \le \beta$

$\alpha + \beta = 4 + \delta$

$a + b = 4$

$\delta > 0$

$$\frac{|x|^{\alpha} \cdot |y|^{\beta}}{x^4 + y^4}$$

$$\|$$

$(x, y) \mapsto x \to |x|^{\delta}$

$$\frac{|x|^{a} \cdot |y|^{b}}{x^4 + y^4} \left( |x|^{\alpha - a} \cdot |y|^{\beta - b} \right)$$

limited $\qquad \downarrow 0$

$x$

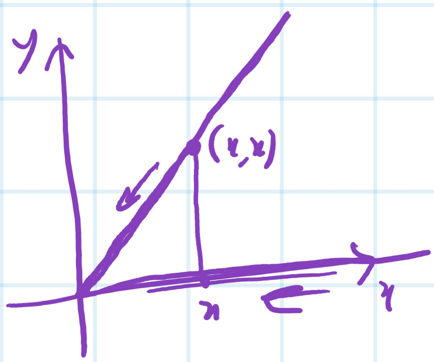
$|x|^{\delta}$

---

$$\lim_{\substack{(x,y) \to (0,0) \\ (x,y) \in \Gamma}} \frac{|x|^{\alpha} \cdot |y|^{\beta}}{x^4 + y^4} = \lim_{x \to 0^+} \frac{|x|^{\alpha + \beta}}{2x^4} =$$

$\Gamma = \{(x, y) \mid x = y, \; x > 0, \; y > 0\}$

$(x, 0)$



$$\lim_{(x,y)\to(0,0)} \frac{x^{100}\cdot y^{100}}{x^{102}+y^{102}+(xy)^{60}} = \lim_{(x,y)\to(0,0)} \frac{x^{100}\cdot y^{100}}{\left(x^{102}+y^{102}\right)\left(1+\frac{(xy)^{60}}{x^{102}+y^{102}}\right)}$$

$$\frac{x^{60}\cdot y^{60}}{x^{102}+y^{102}}$$

$$\left(\frac{x^{60}\cdot y^{42}}{x^{102}+y^{102}}\cdot y^{18}\right) \to 0$$

limitada; $y^{18} \to 0$

$$\left(\frac{x^{100}\cdot y^{2}}{x^{102}+y^{102}}\cdot y^{98}\right) \to 0$$

limitada; $y^{98} \to 0$

---

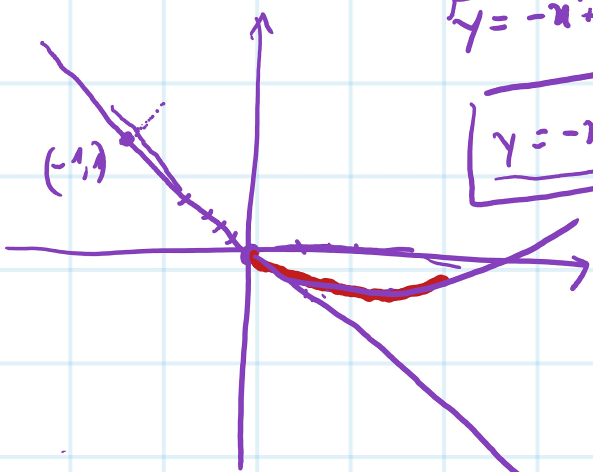
**3**

$$\lim_{(x,y)\to(0,0)} \frac{x^{100}y^{100}}{x^3+y^3}$$

$(x, 0)$

$y = -x + \bullet$

$y = -x + x^{200}$



$$\lim_{\substack{(x,y)\to(0,0)\\ (x,y)\in\Gamma}} \frac{x^{100}\cdot y^{100}}{x^3+y^3} = \lim_{x\to 0^+} \frac{x^{100}\cdot(-x+x^{200})^{100}}{x^3+(-x+x^{200})^3} =$$

$$\Gamma = \{(x,y)\in\mathbb{R}^2 \mid y=-x+x^{200},\ x>0\}$$

$$(x,\ -x+x^{200})$$

$$= \lim_{x\to 0^+} \frac{x^{100}\cdot x^{100}(-1+x^{199})^{100}}{x^3 - x^3 + 3x^2\cdot x^{200} + o(x^{202})}=$$

$$= \lim_{x\to 0^+} \frac{x^{200}}{3x^{202}} = +\infty$$

---

④ $$\lim_{(x,y)\to(0,0)} \frac{xy^5}{x^2+y^8+x^6}$$

$$\frac{xy^6}{x^2+y^8} \to 0 \qquad \lim_{\substack{(u,v)\to(0,0)\\ v>0}} \frac{u\cdot v}{u^2+v^2}\, v^{\frac{1}{2}} = 0$$

$$\frac{x\cdot(y^4)}{x^2+(y^4)^2}\cdot(y^4)^{\frac{1}{2}}$$

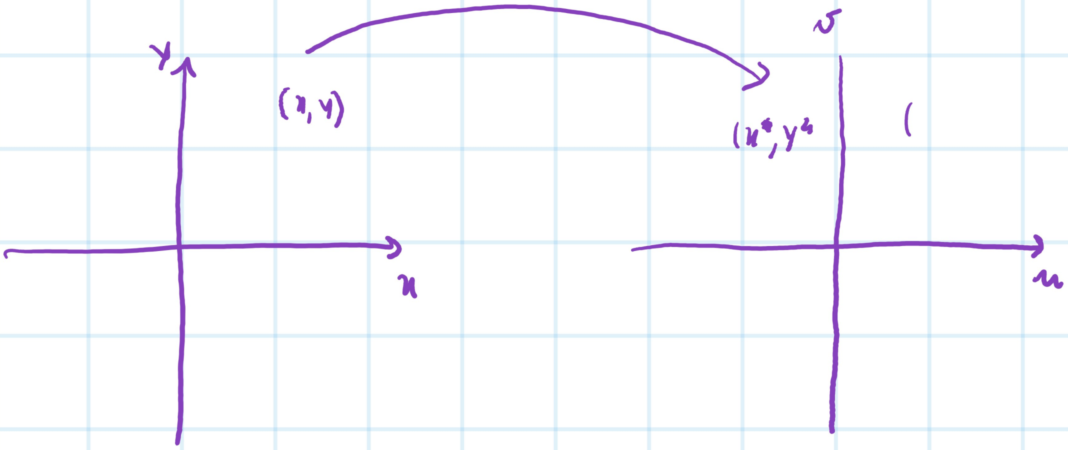
$x = u$

$y^4 = v$

$$\frac{u \cdot v}{u^2 + v^2} \cdot v^{\frac{1}{2}}$$

$u = x$

$v = y^4$




$(u, v) \longrightarrow \dfrac{u\, v^{\frac{3}{2}}}{u^2 + v^2}$

$u(x,y) \quad v(x,y)$

$\phi: (x, y) \mapsto (x, y^4)$

$g(u,v): (u, v) \longmapsto \dfrac{u\, v^{\frac{3}{2}}}{u^2 + v^2}$

$$f(x,y) = g \circ \phi$$

$$f(x,y) = \frac{x y^6}{x^2 + y^8}$$

$$\left|\frac{x y^4}{x^2+y^8}\right| \cdot y^2 \to 0$$

limit

$$\frac{|x|}{\sqrt{x^2+y^8}} \cdot \frac{y^4}{\sqrt{x^2+y^8}} \leq 1 \cdot 1 = 1$$

---

$$\lim_{(x,y)\to(0,0)} \frac{x y^4}{x^2+y^8} \cdot y = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{x^{\alpha} y^{3}}{(x^{30}+y^{18})(x^{10}+y^{10})} \qquad (\alpha \geq 0)$$

$$\lim_{(x,y)\to(0,0)} \frac{x^{\alpha} \cdot y^{3}}{(x^{30}+y^{18})\cdot(x^{10}+y^{10})}$$

$\alpha = 37+\delta$

$y^3 \quad x^{\alpha}$

$\alpha$

$$\frac{x^{25} \cdot y^{3}}{x^{30}+y^{18}} \cdot \frac{x^{10}}{x^{10}+y^{10}}$$

$\alpha > 35$

$\alpha > 35 \quad l = 0$

$\alpha = 35+\delta \quad \delta > 0$

$\delta > 0$

$$\lim_{(x,y)\to(0,0)} \frac{x^{35+\delta} \cdot y^{3}}{(x^{30}+y^{18})(x^{10}+y^{10})} =$$

$$= \lim_{(x,y)\to(0,0)} \underbrace{\frac{x^{25} y^{3}}{x^{30}+y^{18}}}_{\text{limitata}} \cdot \underbrace{\frac{x^{10}}{x^{10}+y^{10}}}_{\leq 1} \cdot \underbrace{x^{\delta}}_{\to 0} = 0$$

$$\left| \frac{x^{25} y^{3}}{x^{30}+y^{18}} \right| = \frac{|x^5|^5 \cdot |y^3|^1}{|x^5|^6 + |y^3|^6}$$

$A = |x^5|$

$B = |y^3|$

$\forall A, B \geq 0$

$$\frac{A^5 B^1}{A^6+B^6}$$

---

$0 \leq \alpha \leq 35$

$$\lim_{(x,y)\to(0,0)} \frac{x^\alpha y^3}{(x^{30}+y^{18})(x^{10}+y^{10})} = \lim_{x\to 0^+} \frac{x^\alpha \cdot (x^{\frac{5}{3}})^3}{(x^{30}+x^{30})(x^{10}+x^{\frac{50}{3}})} =$$

$(x,y) \in \Gamma$

$$= \lim_{x\to 0^+} \frac{x^{5+\alpha}}{2x^{40}(1+o(1))} \qquad 5+\alpha \leq 40$$

$\Gamma = \{(x,y) \,/\, x^5 = y^3, x>0, y>0\}$

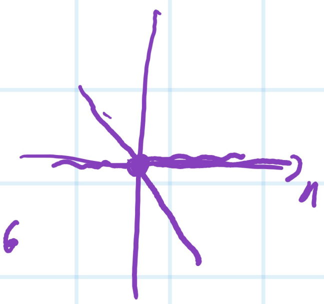
$y = x^{\frac{5}{3}}$

---

$$\lim_{(x,y)\to(0,0)} \frac{x^{100}y^{100}}{x^3+y^3+x^6+y^6}$$

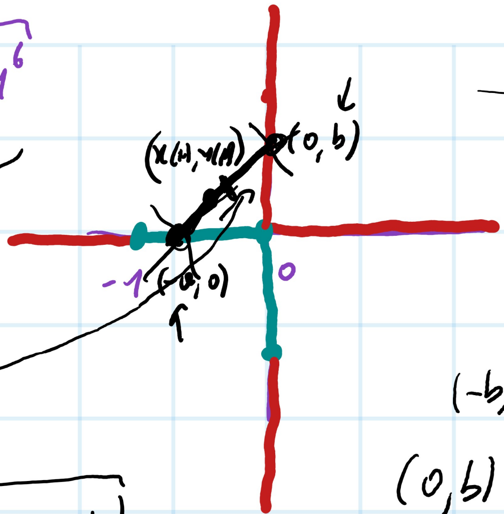
$$\lim_{(x,y)\to(0,0)} \frac{x^{100}\cdot y^{100}}{x^3+y^3+x^6+y^6} =$$

$Y = (\smile) + \bullet$



$x^3 + x^6$

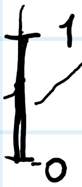
$D(x,y) = x^3 + y^3 + x^6 + y^6$




$f(t) = D(x(t), y(t))$

$(-b, a)$

$(0, b)$




$(-t\cdot a,\ (1-t)b)$

$t \overset{\varphi}{\longmapsto} (-t\cdot a,\ (1-$

D(

$(D\circ\varphi)(t) = D(-ta,\ (1-t)b)$