# A M2- EXE (16-05-2023) LIMITI IN 2 VARIABILI — DIFFERENZIABILITÀ

CALCOLARE I SEGUENTI LIMITI:

$\boxed{1}$ $\lim_{(x,y)\to\infty} x^2+y^2$

$\boxed{2}$ $\lim_{(x,y)\to\infty} x^8+y^8$

$\boxed{3}$ $\lim_{(x,y)\to\infty} x^2y^2$

$\boxed{4}$ $\lim_{(x,y)\to\infty} x^2+y^2+\alpha xy$ $(\alpha\in\mathbb{R})$

$\boxed{5}$ $\lim_{(x,y)\to\infty} \frac{x^6+y^6}{x^2+y^4}$

$\boxed{6}$ $\lim_{(x,y)\to\infty} \frac{xy}{2+x^2y^2}$

$\boxed{7}$ $\lim_{(x,y)\to\infty} \frac{xy^2}{x^4+y^4+xy}$

STUDIARE REGOLARITÀ DI:

$\boxed{8}$ $f(x,y)=\begin{cases}\frac{x^3+y^3}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$

$\boxed{9}$ $f(x,y)=\begin{cases}\frac{xy^\alpha}{x^4+y^4} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$

$\boxed{10}$ $f(x,y)=\begin{cases}\frac{x^3y^\alpha}{\underbrace{x^4+y^{20}}} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$

$\boxed{11}$ $f(x,y)=\begin{cases}\frac{\sin(x)-\sin y}{x-y} & x\neq y\\ \textcircled{?} & \text{altrimenti}\end{cases}$

$$\frac{x^3\cdot y^\alpha}{x^4+y^{20}}$$

$$\frac{x^3y^\alpha}{\boxed{(x^4+y^{20})\sqrt{x^2+y^2}}}$$

$$\lim_{(x,y)\to(0,0)} \frac{\boxed{f(x,y)}-\cancel{\boxed{\text{[illegible]}}}}{\boxed{\sqrt{x^2+y^2}}}$$

$\omega_1^3\cdot\cancel{\omega_2^4}=\omega_2^3\cdot\cancel{\omega_2^5}$

$\omega_1^8=\omega_2^8$

$\omega_1^3=\omega_2^3$ $x^3$

$\omega_1,\dots,\omega_8$

$\boxed{\omega_1^4=\omega_2^4}$

$$\boxed{\lim_{(x,y)\to\infty} (x^2+y^2)^{\alpha} = +\infty}$$

$\alpha > 0$

$\sqrt{x^2+y^2} > M^{\frac{1}{2\alpha}}$

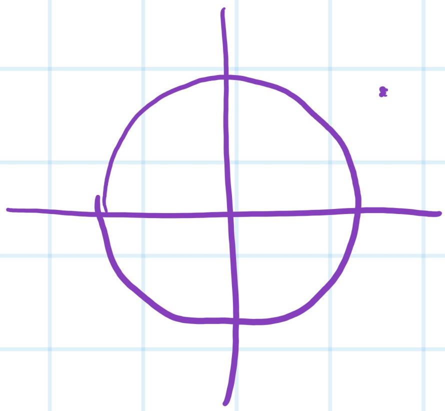
$(x^2+y^2)^{\alpha} > M$

$f(x,y)$

$\forall M > 0 \quad \exists R > 0 \quad t.c. \quad \underbrace{\sqrt{x^2+y^2}} > R \implies x^2+y^2 > M$

$R = \sqrt{M}$

BASTA PRENDERE



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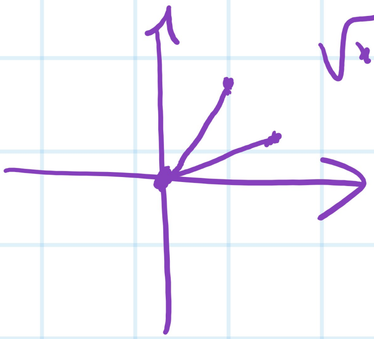
$$\lim_{(x,y)\to\infty} \underbrace{x^8+y^8}$$

$$\underbrace{(x^2+y^2)^4}$$



$\sqrt{x^2+y^2}$

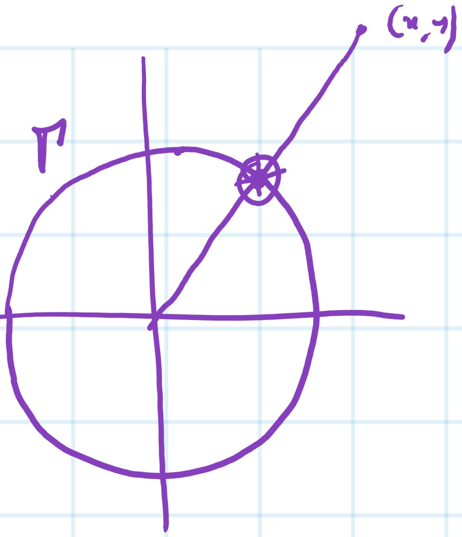
$$x^8+y^8 = \underbrace{\boxed{\frac{x^8+y^8}{(x^2+y^2)^4}}}_{\uparrow} \cdot \overbrace{(x^2+y^2)^4}^{+\infty}$$

$\to g(x,y)$

$$g(\lambda x, \lambda y) = \frac{\cancel{\lambda^8}(x^8+y^8)}{\cancel{\lambda^8}(x^2+y^2)^4} = g(x,y)$$

$$g(x,y) = \frac{x^8+y^8}{(x^2+y^2)^4}$$

$$C = \text{MIN}\{g(x,y) \mid (x,y) \in \Gamma\}$$



$$\boxed{C > 0}$$

$$x^8+y^8 = \frac{x^8+y^8}{(x^2+y^2)^4} \cdot (x^2+y^2)^4 \geq C\underbrace{(x^2+y^2)^4}_{\downarrow +\infty}$$

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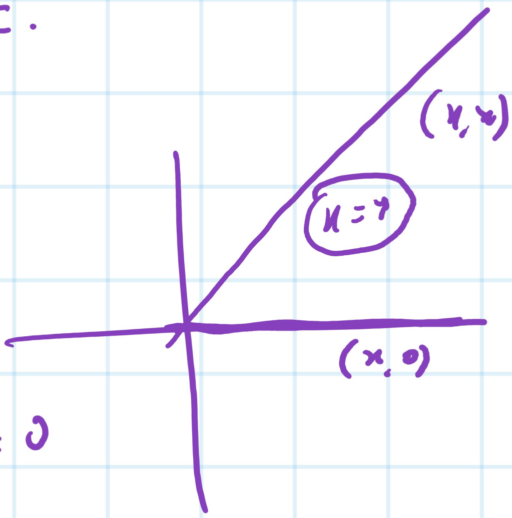
$$\lim_{(x,y)\to\infty} \underbrace{x^2y^2} = N.E.$$



$$\lim_{x\to+\infty} x^4 = +\infty$$

$$\lim_{x\to+\infty} (x^2 \cdot 0^2) = \lim_{x\to+\infty} 0 = 0$$

$\alpha \in \mathbb{R}$

$$\lim_{(x,y)\to\infty} \left|x^2+y^2+\alpha xy\right| = ?$$

$|\alpha|$

$$\boxed{x^2+y^2+\alpha xy} = (x^2+y^2)\left(1+\alpha\,\boxed{\frac{xy}{x^2+y^2}}\right) \geq \underbrace{(1-c)}\cdot\underbrace{(x^2+y^2)}_{\to +\infty}$$

$| \quad | \leq \frac{1}{2}$

$\boxed{|\alpha|<2}$

$|x^2+y^2| \geq |2xy|$ ?

$x^2+y^2 \geq 2|x||y|$ ?

$x^2+y^2-2|x||y| \geq 0$ ?

$(|x|-|y|)^2 \geq 0$ ?

$|\bullet| \leq c < 1$

$(1-c) \leq |1 + \bigcirc|$

$\boxed{\begin{matrix}\alpha = 2 \\ -2\end{matrix}}$ lim M.F.

$f(x,y) = x^2+y^2+2xy = (x+y)^2$

$-2xy = (x-y)^2$

$\boxed{\alpha>2}$

$x^2+y^2+\alpha xy$

$y=x \quad x^2+x^2+\alpha x^2 = (2+\alpha)x^2$

$y=-x$ $\qquad x^2+y^2-\alpha x^2 =$

$=(2-\alpha)x^2$

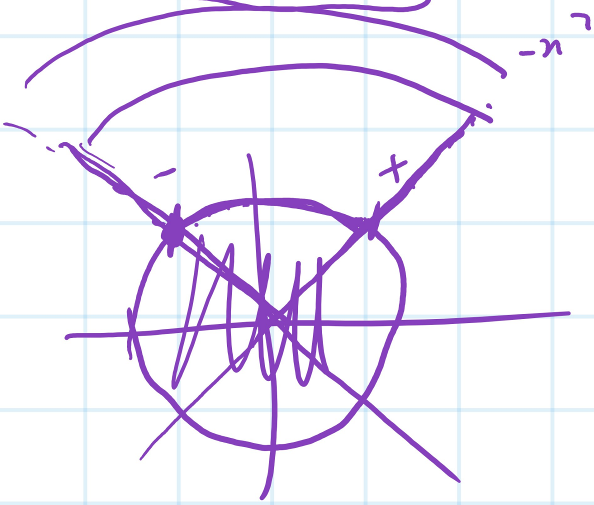
$|\alpha|<2 \qquad \lim = +\infty$

$\left.\begin{array}{l}|\alpha|=2\\ \boxed{|\alpha|>2}\end{array}\right\} \to \lim$ N.E.

$x^2-2xy+y^2$

$x^2-y^2 \qquad \boxed{x^2+y^2+4xy} \qquad y=x$

$-x^2$



$$\lim_{(x,y)\to\infty} \frac{x^6+y^6}{x^4+y^2} =$$

$v=\frac{1}{x} \quad w=\frac{1}{y}$

$$\lim_{(x,y)\to\infty} f(x,y) = \lim_{(u,v)\to(0,0)} f\left(\frac{1}{v},\frac{1}{w}\right)$$

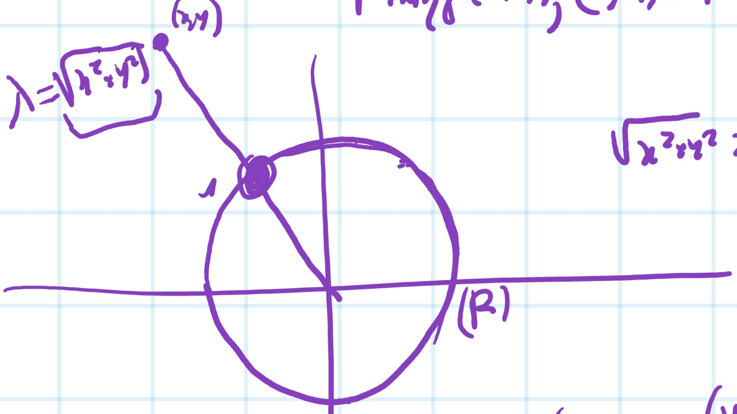
NO

$$\frac{x^6+y^6}{x^4+y^2} = \underbrace{\boxed{\frac{x^6+y^6}{(x^4+y^4)}}}_{\to +\infty} \; \underbrace{\boxed{\frac{(x^4+y^4)}{x^4+y^2}}}_{\geq C > 0}$$

$g(x,y)$

$$g(\lambda x, \lambda y) = \frac{\lambda^6(x^6+y^6)}{\lambda^4(x^4+y^4)} = \lambda^2 \cdot g(x,y)$$

$$\mathrm{MIN}\{g(x,y), (x,y)\in\Gamma\} = M > 0$$



$$\sqrt{x^2+y^2} \geq R$$

$$g(x,y) \geq \underline{(\sqrt{x^2+y^2})^2 \cdot M} \to +\infty$$

$$\boxed{g(x,y)} = g\left(\sqrt{x^2+y^2}\cdot\frac{x}{\sqrt{x^2+y^2}}, \sqrt{x^2+y^2}\cdot\frac{y}{\sqrt{x^2+y^2}}\right) =$$

$$= (\sqrt{x^2+y^2})^2\, g\underbrace{\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)} \geq \underbrace{(\sqrt{x^2+y^2})^2 \cdot C}_{\downarrow \; +\infty}$$

$g(x,y)$ POS. OMO. DI GRADO $\alpha > 0$

$$g(x,y) = g\left(\boxed{\sqrt{x^2+y^2}} \cdot \frac{x}{\sqrt{x^2+y^2}} , \boxed{\sqrt{x^2+y^2}} \cdot \frac{y}{\sqrt{x^2+y^2}}\right) =$$

$$= \left(\sqrt{x^2+y^2}\right)^{\alpha} g\Big(\underbrace{\frac{x}{\sqrt{x^2+y^2}} , \frac{y}{\sqrt{x^2+y^2}}}_{\hookrightarrow \in \Gamma}\Big) \geq \underbrace{\left(\sqrt{x^2+y^2}\right)^{\alpha} \cdot M}_{\downarrow +\infty}$$

$$\frac{x^4+y^4}{x^4+y^2} \geq C > 0 \ (?)$$

Palla di raggio R

su $\mathbb{R}^2 - B_R$

$\alpha + \beta = 4$

$$\underbrace{2(x^4+y^4)} \overset{?}{\geq} C(x^4+y^2)$$

$$\underbrace{\frac{x^{\alpha} \cdot y^{\beta}}{x^4+y^4}}_{\leq} \leq$$

$$x^4+y^4 + \boxed{x^4+y^4} \geq$$

$$\geq x^4 + \underbrace{y^4 + x^2 \cdot y^2} = x^4 + y^2 \overline{(y^2+x^2)} \geq x^4 + y^2 \cdot \boxed{1}$$

$\boxed{\alpha+\beta=4}$

$$1 \geq \left(\frac{x}{\sqrt[4]{x^4+y^4}}\right)^{\alpha} \cdot \left(\frac{y}{\sqrt[4]{x^4+y^4}}\right)^{\beta} = \frac{x^{\alpha}y^{\beta}}{x^4+y^4}$$

$$\frac{x^{\alpha}\cdot y^{\beta}}{x^4+y^4} \leq 1 \qquad \alpha+\beta=4$$

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$$\frac{x^4+y^4}{x^4+y^2} \geq C > 1$$ FUORI DA PALLA

$$\boxed{2(x^4+y^4)} = x^4+y^4+\underbrace{x^4+y^4} \geq x^4+\underbrace{y^4+x^2y^2} =$$

$$= x^4+y^2\underbrace{(y^2+x^2)} \geq \boxed{x^4+y^2}$$

fuori da Palla raggio 1

$$\frac{x^4+y^4}{x^4+y^2} \geq \boxed{\frac{1}{2}}$$

$\alpha > \beta > 0$   $\exists R > 0$ t.c. $\|(x,y)\| \geq R$ ALLORA

$$|x|^{\overset{\downarrow}{\alpha}} + |y|^{\overset{\downarrow}{\alpha}} > |x|^{\overset{\downarrow}{\alpha}} + |y|^{\overset{\downarrow}{\beta}}$$

$$|x|^{\alpha} + |y|^{\alpha} \geq |y|^{\beta} |x|^{\alpha-\beta}$$

$$\frac{|y|^{\beta} \cdot |x|^{\alpha-\beta}}{|x|^{\alpha} + |y|^{\alpha}}$$

$$2(|x|^{\alpha} + |y|^{\alpha}) = |x|^{\alpha} + |y|^{\alpha} + \underbrace{|x|^{\alpha} + |y|^{\alpha}} \geq$$

$$= |x|^{\alpha} + \underbrace{|y|^{\alpha} + |y|^{\beta} \cdot |x|^{\alpha-\beta}} =$$

$$\underbrace{\frac{|y|^{\beta} \cdot |x|^{\alpha-\beta}}{|x|^{\alpha} + |y|^{\alpha}}} = \frac{|y|^{\beta}}{\left(\left(|x|^{\alpha} + |y|^{\alpha}\right)^{\frac{1}{\alpha}}\right)^{\beta}} \cdot \frac{|x|^{\alpha-\beta}}{\left(\left(|x|^{\alpha} + |y|^{\alpha}\right)^{\frac{1}{\alpha}}\right)^{\alpha-\beta}} =$$

$$\boxed{\frac{\beta}{\alpha}} \qquad \boxed{1 - \frac{\beta}{\alpha}}$$

$$= \left(\underbrace{\frac{|y|}{\left(|x|^{\alpha} + |y|^{\alpha}\right)^{\frac{1}{\alpha}}}}_{\leq 1}\right)^{\beta} \cdot \left(\underbrace{\frac{|x|}{\left(|x|^{\alpha} + |y|^{\alpha}\right)^{\frac{1}{\alpha}}}}_{\leq 1}\right)^{\alpha-\beta} \leq 1$$

$$= |x|^{\alpha} + |y|^{\beta} (\underbrace{(|y|^{\alpha-\beta} + |x|^{\alpha-\beta})}_{\boxed{\geq 1}} \geq |x|^{\alpha} + |y|^{\beta}$$

$\boxed{\exists R > 0 \text{ T.C.}}$

$x^8+y^8 \geq x^8+y^4$ fuori dalla Palla

$$\boxed{\frac{x^8+y^4}{x^8+y^8}} \leq 1$$

$x^8+y^4 = O(x^8+y^8)$ per $(x,y)\to\infty$

$\alpha>\beta>0$

$\underline{|x|^\alpha+|y|^\beta} = O((|x|^\alpha+|y|^\alpha)$ per $(x,y)\to\infty$

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$$\lim_{(x,y)\to\infty} \frac{xy}{2+x^2y^2} =$$

$$f(x,y)=\begin{cases}\frac{x^3+y^3}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

$$\underbrace{\frac{x^3+y^3}{x^2+y^2}} = \overbrace{\underbrace{\boxed{\frac{x^2}{x^2+y^2}}\cdot \underset{\downarrow \atop 0}{x}} + \underbrace{\boxed{\frac{y^2}{x^2+y^2}}\cdot \underset{\downarrow \atop 0}{y}}} \to 0$$

$v=(v_1,v_2)$

$$\partial_v f(0,0) = \lim_{t\to 0}\frac{\overbrace{f(tv_1,tv_2)} - \overbrace{f(0,0)}^{0}}{t} =$$

$$= \lim_{t\to 0}\frac{\frac{\cancel{t^3}(v_1^3+v_2^3)}{\cancel{t^2}(v_1^2+v_2^2)} - 0}{\cancel{t}} = \boxed{\frac{v_1^3+v_2^3}{v_1^2+v_2^2}}$$

$$\partial_x = \partial_{(1,0)}$$

$$\boxed{\begin{array}{c}\partial_x f(0,0)=1\\ \partial_y f(0,0)=1\end{array}}$$

$v=(v_1,v_2)$ $$\partial_v f(0,0) = \langle v, \nabla f(0,0)\rangle = v_1\cdot 1 + v_2\cdot 1 = \boxed{v_1+v_2}$$

$$\overbrace{f(0,0)}^{0} + \overbrace{f_y(0,0)}^{1}(x-0) + \overbrace{f_y(0,0)}^{1}(y-0)$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - \overset{\text{P. Tange}}{\boxed{\phantom{xxxxxx}}}}{\sqrt{x^2+y^2}} = 0 \ (?)$$

$$\lim_{(x,y)\to(0,0)} \frac{\frac{x^3+y^3}{x^2+y^2} - (x+y)}{\sqrt{x^2+y^2}} \overset{?}{\neq} 0$$

$$\lim_{(x,y)\to(0,0)} \frac{\cancel{x^3}+\cancel{y^3}-\cancel{x^3}-x^2y-xy^2-\cancel{y^3}}{(x^2+y^2)\sqrt{x^2+y^2}} =$$

$$\lim_{(x,y)\to(0,0)} \frac{\overbrace{-xy(x+y)}}{\underbrace{(x^2+y^2)\sqrt{x^2+y^2}}_{?}} =$$

$$\boxed{x=y} \quad \lim_{x\to 0^+} \frac{-2\cancel{x^3}}{2\sqrt{2}\cdot\cancel{x^3}} \neq 0$$