A.M. 2 - LEZ. 1 - INTEGRALE DI RIEMANN (1)

1) ESEMPIO $\int_0^1 x^2\,dx$

2) DEF. PARTIZIONE

3) DEF. SOMME RIEMAN

4) ORDINAMENTO PARTIZIONI E MONOTONIA SOMME RIEMANN

5) DEF. INTEGRALE RIEMANN.

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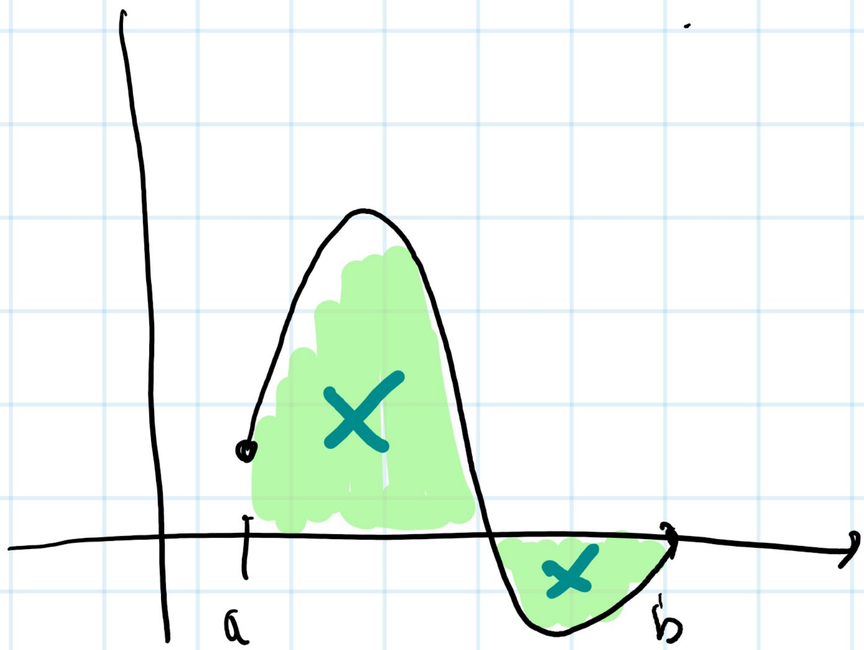
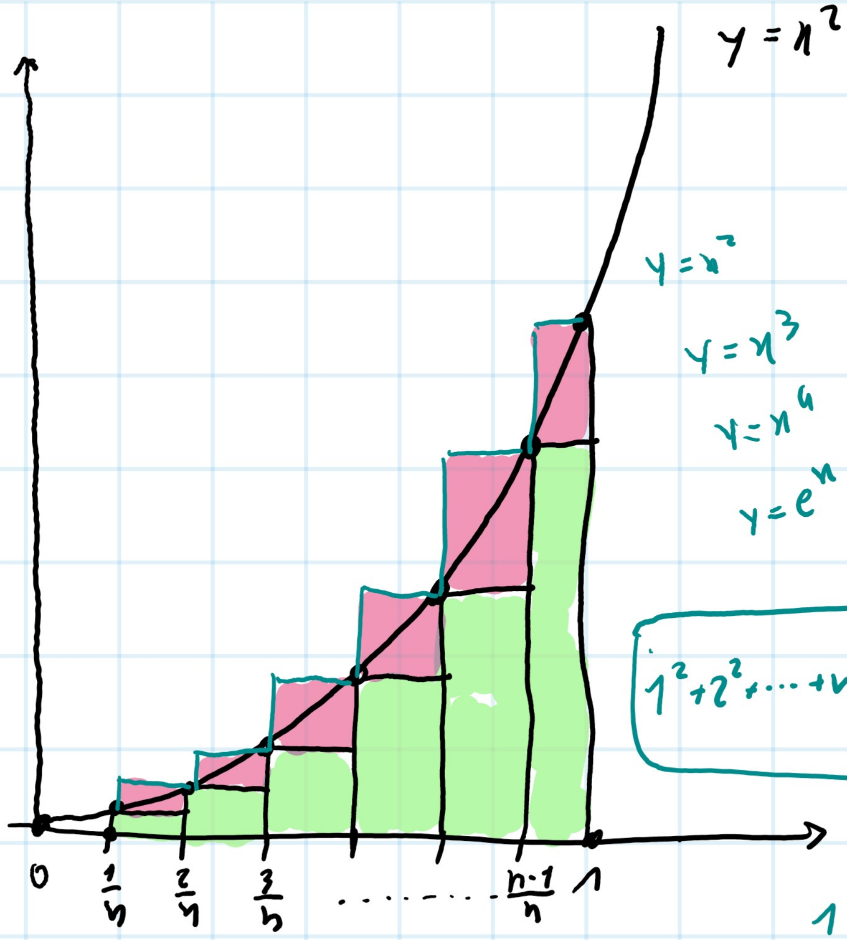
6) CONDIZ. EQUIV. ALL'INTEGRABILITÀ

7) $R([a,b])$ è SP. VETT.



$n \in \mathbb{N}$



$y = x^2$

$y = x^3$

$y = x^4$

$y = e^x \leftarrow$

$$\boxed{1^2+2^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}}$$

$$1+2+\dots+n = \frac{n(n+1)}{2}$$

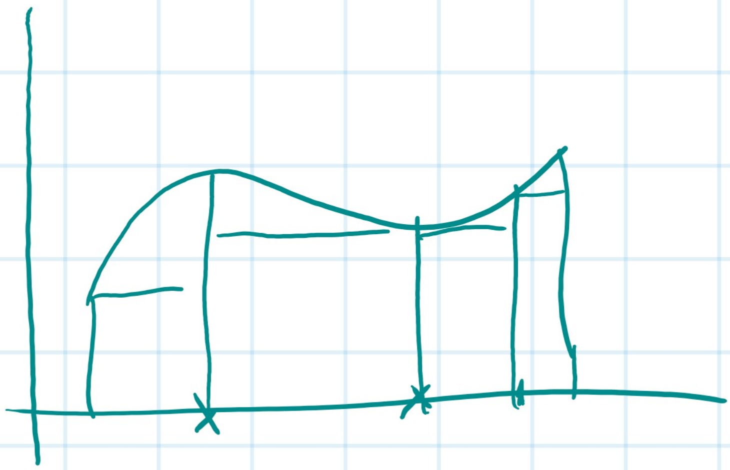
$$s_n = \overbrace{\sum_{k=1}^{n} \frac{1}{n}\cdot\left(\frac{k-1}{n}\right)^2} = \frac{1}{n^3}\cdot\sum_{k=1}^{n}(k-1)^2 = \frac{1}{n^3}\underbrace{\left(0^2+1^2+2^2+\dots+(n-1)^2\right)}$$

$$\boxed{I_k = \left[\frac{k-1}{n}, \frac{k}{n}\right]}$$

$$\boxed{S_n} = \sum_{k=1}^{n}\frac{1}{n}\cdot\left(\frac{k}{n}\right)^2 = \frac{1}{n^3}\left(1^2+2^2+3^2+\dots+n^2\right) = \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{1}{6}\left(2+\frac{2}{n}+\frac{1}{n}+\frac{1}{n^2}\right) = \frac{1}{3}+\underbrace{\frac{1}{2n}+\frac{1}{6n^2}}$$

$$\boxed{s_n} = \frac{1}{n^3}\cdot\frac{\overbrace{(n-1)}\cdot\overbrace{n}\cdot\overbrace{(2n-1)}}{6} = \frac{1}{6}\cdot\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) = \frac{1}{6}\left(2-\frac{2}{n}-\frac{1}{n}+\frac{1}{n^2}\right) = \frac{1}{3}\underbrace{-\frac{1}{2n}+\frac{1}{6n^2}}$$



$\boxed{\text{DEF. 1}}$ DATO $[a,b] \subset \mathbb{R}$ DIREMO CHE

$P = \{x_0, x_1, x_2, \dots, x_n\}$ È PARTIZIONE DI $[a,b]$

SE $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$

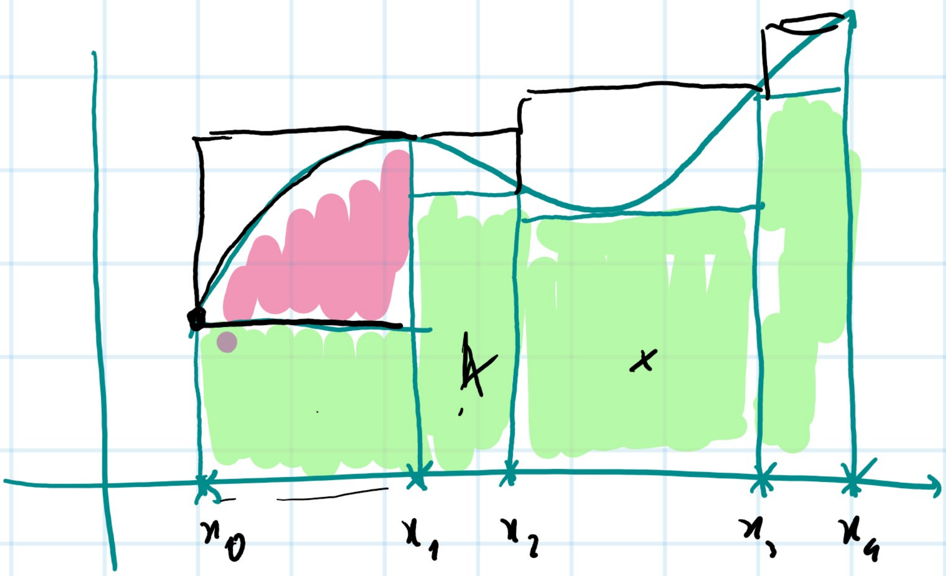
$\boxed{\text{DEF. 2}}$ DATI $[a,b] \subset \mathbb{R}$ ED $f : [a,b] \to \mathbb{R}$ <u>LIMITATA</u>

E DATA $P = \{x_0, x_1, \dots, x_n\}$ PART. DI $[a,b]$

DEFINIAMO

$$s(f,P) = \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot \underbrace{\inf\{f(x) \mid x \in [x_{i-1}, x_i]\}}_{m_i}$$

$$S(f,P) = \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot \underbrace{\sup\{f(x) \mid x \in [x_{i-1}, x_i]\}}_{M_i}$$



$$\underbrace{(x_1-x_0)\cdot \inf\{f(x) \mid x\in[x_0,x_1]\}}_{s\,m_1} + \underbrace{(x_2-x_1)\cdot \inf\{f(x) \mid x\in[x_1,x_2]\}}\cdots\cdots$$

$\boxed{\text{DEF. 3}}$ DATI $[a,b]\subset\mathbb{R}$ $\quad f:[a,b]\to\mathbb{R}$ LIMITATA.

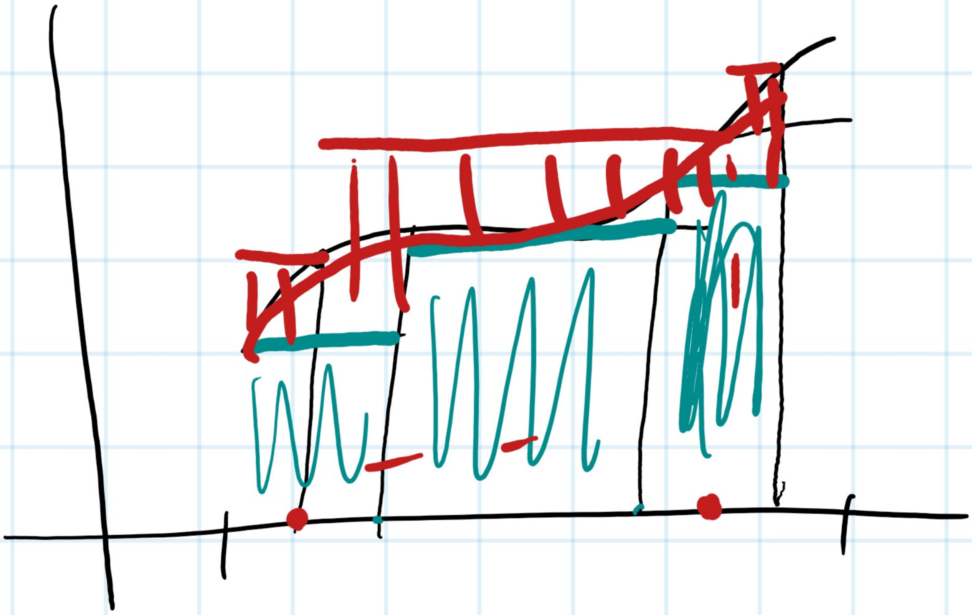
DEFINIAMO

$$\to \int^{-}_{[a,b]} f = \sup\{s(f,P) \mid P \text{ PART. DI } [a,b]\}$$

$$\to \int^{+}_{[a,b]} f = \inf\{S(f,P) \mid P \text{ PART. DI } [a,b]\}$$

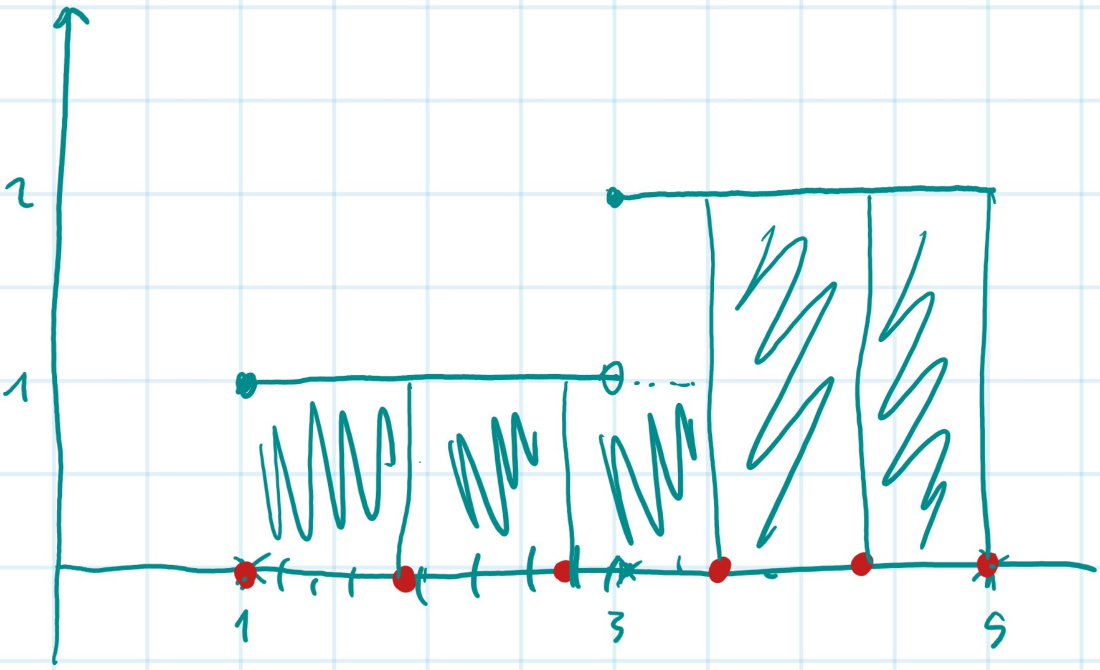
SE $\displaystyle\int^{+}_{[a,b]} f = \int^{-}_{[a,b]} f$ ALLORA DICO CHE $\underline{f\in\mathcal{R}([a,b])}$

E CHE $\displaystyle\int_a^b f(x)\,dx = \boxed{}$



$$\boxed{P_1 = \{1, 3, 5\}}$$

$$s(f, P) = 2 \cdot 1 + 2 \cdot 2 = 6$$



DEF. DATO $[a,b]$ E DATE P_1, P_2 PART. DI $[a,b]$

DIREMO CHE P_1 È PIÙ FINE DI P_2 SE $P_1 \supset P_2$

$\boxed{\text{PROP.}}$ DATI $[a,b] \subset \mathbb{R}$, $f:[a,b] \to \mathbb{R}$ LIMITATA E

P_1, P_2 PART. DI $[a,b]$ T.C. P_1 È PIÙ FINE D P_2

ALLORA

$$s(f,P_2) \leq s(f,P_1) \leq S(f,P_1) \leq S(f,P_2)$$

$\boxed{\text{DIM}}$ MOSTRIAMO $s(f,P_2) \leq s(f,P_1)$

QUANDO

$$P_2 = \{x_0, x_1, \dots, x_{k-1}, x_k, \dots, x_n\}$$

$m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i)\}$

$$P_1 = \{x_0, x_1, \dots, x_{k-1}, \bar{x}, x_k, \dots, x_n\}$$

$$s(f,P_2) = (x_1-x_0)\cdot m_1 + (x_2-x_1)m_2 + \dots + (x_k-x_{k-1})\cdot m_k + \dots \dots (x_n-x_{n-1})m_n$$

$$s(f,P_1) = (x_1-x_0)m_1 + (x_2-x_1)m_2 + \dots + (\bar{x}-x_{k-1})m_k' + (x_k-\bar{x})m_k'' + \dots + (x_n-x_{n-1})m_n$$

$$\geq (\bar{x}-x_{k-1})m_k + (x_k-\bar{x})m_k$$

$$m_k = \inf\{f(x) \mid x \in [x_{k-1}, x_k)\}$$

$$= (\cancel{\bar{x}} - x_{k-1} + x_k - \cancel{\bar{x}})m_k$$

$$\underline{m_k'} = \inf\{f(x) \mid x \in [x_{k-1}, \bar{x})\} \geq m_k$$

$$\underline{m_k''} = \inf\{f(x) \mid x \in [\bar{x}, x_k)\} \geq m_k$$

PROP. DATI $[a,b] \subset \mathbb{R}$, $f: [a,b] \to \mathbb{R}$ LIMITATA E P_1, P_2 PART. DI $[a,b]$. ALLORA

$$\underline{s(f,P_1)} \leq S(f,P_2)$$

DIM (OVVIO SE $P_1 = P_2$)

ALTRIMENTI

$$\underline{P} = P_1 \cup P_2$$

$$s(f,P_1) \leq s(f,P) \leq S(f,P) \leq S(f,P_2)$$

TEO DATI $[a,b] \subset \mathbb{R}$ $f: [a,b] \to \mathbb{R}$ LIMITATA. ALLORA

È EQ. AFFERMARE CHE

1) $f \in \mathcal{R}([a,b])$

$\to$ 2) $\forall \varepsilon > 0 \ \exists P$ PART. DI $[a,b]$ t.c. $S(f,P) - s(f,P) < \varepsilon$