A.M.2 - LEZ. 2 - INTEGRALE DI RIEMANN (2)

1) CONDIZIONI EQUIVALENTI ALL'INTEGRABILITÀ

2) $\mathcal{R}([a,b])$ È SP. VETT. E $\int$ È LINEARE.

3) $f,g \in \mathcal{R}([a,b]) \Rightarrow f^+, f^-, |f|, \max\{f,g\}, \min\{f,g\} \in \mathcal{R}([a,b])$

4) MONOTONIA

5) $|\int f| \leq \int |f|$

6) ADDITIVITÀ SU INTERVALLI

7) $\int$ ORIENTATO DEF. E ADDITIVITÀ

8) CONDIZ. SUFFICIENTI

TEO 1 DATI $[a,b] \subset \mathbb{R}$ $f:[a,b] \to \mathbb{R}$ LIMITATA, ALLORA È EQ. DIRE CHE:

1) $f \in \mathcal{R}([a,b])$

2) $\forall \varepsilon > 0$ $\exists P$ PART. DI $[a,b]$ t.c. $S(f,P) - s(f,P) < \varepsilon$

3) $\forall \varepsilon > 0$ $\exists P$ PART. DI $[a,b]$ t.c. $\sum_{i=1}^{n} (x_i - x_{i-1}) \cdot osc(f,[x_{i-1},x_i]) < \varepsilon$

$P = \{x_0, x_1, \dots, x_n\}$

N.B.

$osc(f,B) = \sup_{x \in B} f - \inf_{x \in B} f$

DIM

$$S(f,P) - s(f,P) = \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot \sup\{f(x) \mid x \in [x_{i-1},x_i]\} - \dots \inf \dots$$

$$= \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot osc(f,[x_{i-1},x_i])$$

(1) ⇒ (2) $\forall \varepsilon > 0$ $\exists P_1$ t.c. $s(f,P_1) > \underline{\int} f - \frac{\varepsilon}{2} = \int_a^b f(x)\,dx - \frac{\varepsilon}{2}$

$\exists P_2$ t.c. $S(f,P_2) < \overline{\int} f + \frac{\varepsilon}{2} = \int_a^b f(x)\,dx + \frac{\varepsilon}{2}$

$$\int_a^b f(x)\,dx - \frac{\varepsilon}{2} < s(f,P_1) \le \int_a^b f(x)\,dx \le S(f,P_2) < \int_a^b f(x)\,dx + \frac{\varepsilon}{2}$$

$P = P_1 \cup P_2$

$$s(f,P_1) \le s(f,P) \le S(f,P) \le S(f,P_2)$$

$$\boxed{S(f,P) - s(f,P) < \varepsilon}$$

(2) ⇒ (1) $\forall \varepsilon > 0$ $\exists P$ t.c. $S(f,P) - s(f,P) < \varepsilon$

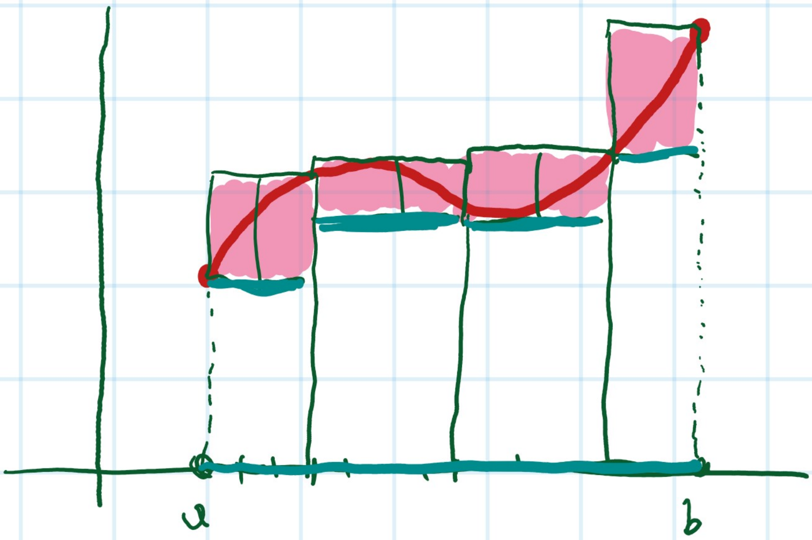
$$\underbrace{s(f,\mathcal{P}) \le \underbrace{\int^- f \le \int^+ f} \le S(f,\mathcal{P})}$$

$$\boxed{0 \le \int^+ f - \int^- f \le \varepsilon \qquad \forall \varepsilon > 0}$$

$$\underbrace{\int^+ f} - \underbrace{\int^- f} = 0$$

$\boxed{T.2}$ DATI $[a,b] \subset \mathbb{R}$ E $f \in C([a,b])$ ALLORA $f \in \mathcal{R}([a,b])$

$\boxed{\text{DIM.}}$ (?) $\forall \varepsilon > 0$ $\exists \mathcal{P}$ $\underline{S(f,\mathcal{P}) - s(f,\mathcal{P}) < \varepsilon}$ (??)



$\boxed{f \text{ È U.C.}}$ PRENDO $\delta > 0$ T.C. $\forall x,y \in [a,b]$ $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a}$

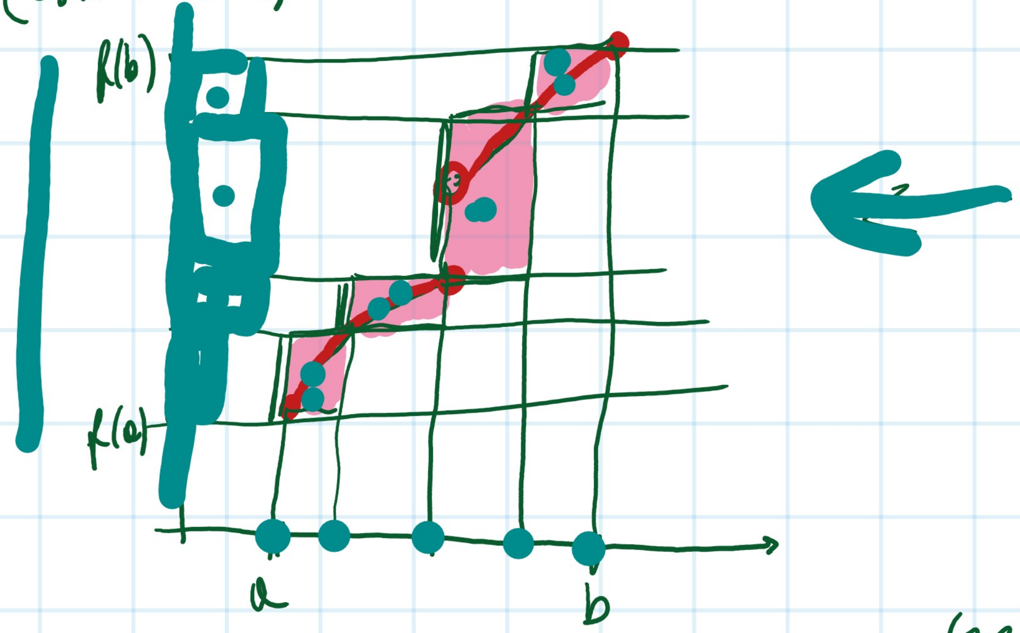
PRENDO $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ DI $[a,b]$ T.C. $\boxed{\forall i = 1, \dots, n \quad |x_i - x_{i-1}| < \delta}$

$$\underbrace{S(f,\mathcal{P}) - s(f,\mathcal{P})} = \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot \underbrace{\text{osc}(f, \overline{[x_{i-1}, x_i]})} \le$$

$$\leq \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot \frac{\varepsilon}{b-a} = \frac{\varepsilon}{b-a} \sum_{i=1}^{n} (x_i - x_{i-1}) = \frac{\varepsilon}{b-a} \cdot (b-a) = \varepsilon$$

$\boxed{\text{T.3}}$ DATI $[a,b] \subset \mathbb{R}$ $f:[a,b] \to \mathbb{R}$ MONOTONA, ALLORA $f \in R([a,b])$

$\boxed{\text{DIM}}$ (caso crescente)

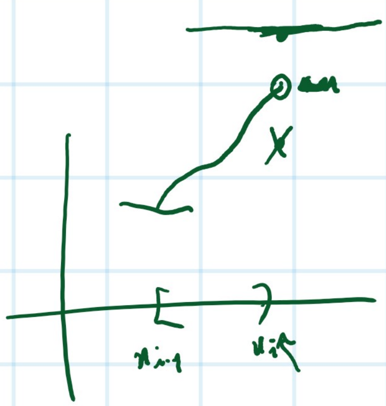


(?) $\forall \varepsilon > 0 \quad \exists P = \{x_0, x_1, \dots, x_n\}$ t.c. $S(f,P) - s(f,P) < \varepsilon$ (??)

PRENDO P t.c. $\forall i = 1, \dots, n \quad |x_i - x_{i-1}| < \frac{\varepsilon}{f(b)-f(a)}$

$$\underline{S(f,P) - s(f,P)} = \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot \operatorname{osc}(f, [x_{i-1}, x_i]) \leq$$

$$\leq \sum_{i=1}^{n} \frac{\varepsilon}{f(b)-f(a)} \cdot \operatorname{osc}(f, [x_{i-1}, x_i]) =$$



$$= \frac{\varepsilon}{f(b)-f(a)} \cdot \sum_{i=1}^{n} \overbrace{\operatorname{osc}(f, [x_{i-1}, x_i])}^{\leq f(x_i) - f(x_{i-1})} \leq$$

$$\leq \frac{\varepsilon}{f(b)-f(a)} \cdot \underbrace{\sum_{i=1}^{n} \overbrace{\left(f(x_i) - f(x_{i-1})\right)}^{-f(x_{i-1}) + f(x_i)}} =$$

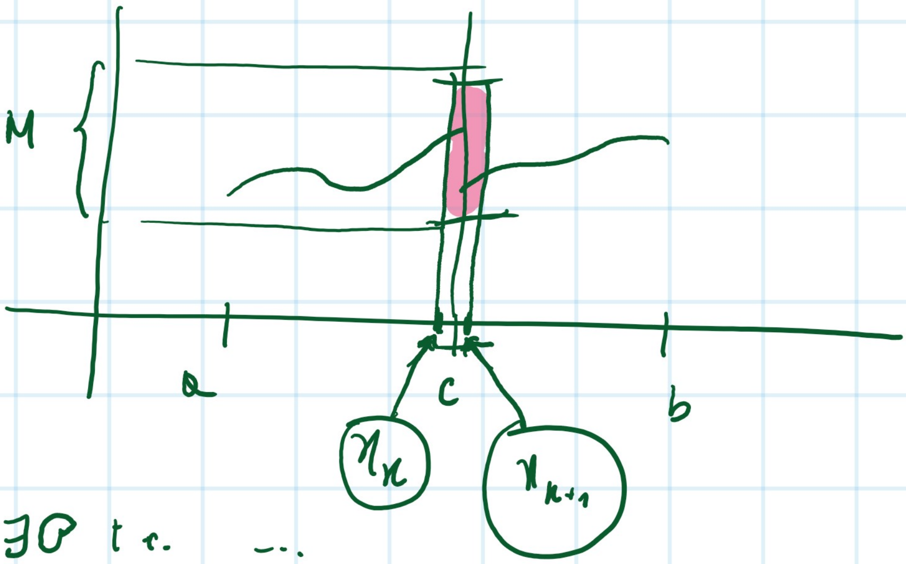
$$= \frac{\varepsilon}{f(b)-f(a)} \cdot \left(\overbrace{-f(x_0)}^{-f(a)} + \cancel{f(x_1)} - \cancel{f(x_1)} + \cancel{f(x_2)} - \cancel{\dots} \quad \dots \quad - \cancel{f(x_{n-1})} + \overbrace{f(x_n)}^{f(b)} \right) =$$

$$= \frac{\varepsilon}{\cancel{f(b)-f(a)}} \cdot \left(\cancel{f(b)-f(a)} \right) = \varepsilon$$

ALTRI CASI

ES. $f: [a,b] \to \mathbb{R}$ LIMITATA CONTINUA IN TUTTI I PUNTI TRANNE $c \in (a,b)$

$$\boxed{M = \operatorname{osc}(f, [a,b])}$$



$\forall \varepsilon > 0 \ \exists \sigma$ t.c. ...

$$x_{k+1} - x_k < \frac{\varepsilon}{2M}$$

ES. PER CASA DATA $f:[a,b]\to\mathbb{R}$ LIMITATA E CONTINUA SU

$[a,b]-A$

$\underline{A=\{a_1, a_2, \dots, a_n, \dots\}}$ t.c. $a_n \to \ell \in [a,b]$

MOSTRARE CHE $f \in \mathcal{R}([a,b])$

TEO. DATI $[a,b]\subset\mathbb{R}$ E $f,g\in\mathcal{R}([a,b])$ E $\alpha\in\mathbb{R}$ ALLORA

→ 1) $\underline{\alpha f}\in\mathcal{R}([a,b])$ E $\displaystyle\int_a^b \alpha f(x)\,dx = \underbrace{\alpha\int_a^b f(x)\,dx}$

2) $f+g\in\mathcal{R}([a,b])$ E $\displaystyle\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

DIM

(1) $\boxed{\alpha>0}$

$$s(\alpha f, \mathcal{P}) = \sum_{i=1}^{n}(x_i - x_{i-1})\cdot \inf\{\alpha f(x) \mid x\in[x_{i-1}, x_i)\} =$$

$$= \alpha \sum_{i=1}^{n}(x_i - x_{i-1}) \inf\{f(x) \mid x\in[x_{i-1}, x_i)\} =$$

$$= \alpha \cdot s(f, \mathcal{P})$$

$$S(\alpha f, P) = \alpha S(f, P)$$

$$\alpha \int_a^b f(x)\,dx$$

$$\boxed{\overline{\int} \alpha f} = \sup\{s(\alpha f, P) \mid P\} = \sup\{\alpha\, s(f,P) \mid P\} = \alpha \cdot \overline{\sup\{s(f,P) \mid P\}} =$$

$$= \alpha \cdot \overline{\int} f = \boxed{\alpha \int_a^b f(x)\,dx}$$

$$\boxed{\int^{+} \alpha f} = \inf\{S(\alpha f, P) \mid P\} = \inf\{\alpha S(f,P) \mid P\} = \alpha\, \overbrace{\inf\{S(f,P) \mid P\}} =$$

$$= \alpha \cdot \int^{+} f = \boxed{\alpha \int_a^b f(x)\,dx}$$

$$\boxed{\alpha = -1}$$

$$A = \{\lambda \in \mathbb{R} \mid \underline{\lambda \le -f(x)\ \forall x \in I_i}\}$$

$$\boxed{s(-f, P)} = \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot \inf\overbrace{\{-f(x) \mid x \in [x_{i-1}, x_i]\}} =$$

$$= \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot \boxed{(-1)} \cdot \sup\underbrace{\{f(x) \mid x \in [x_{i-1}, x_i]\}} =$$

$$B = \{\mu \in \mathbb{R} \mid \overline{\mu \ge f(x)\ \forall x \in I_i}\}$$

$$\mu \in B \Leftrightarrow -\mu \in A$$

$$\sup \circledcirc = \min B = -\max A = \inf \circledcirc$$

$$\overbrace{(-1)\cdot\sum_{i=1}^{n}(x_i-x_{i-1})\sup\{f(x)\mid x\in[x_{i-1},x_i]\}}=$$

$$=\boxed{-S(f,P)}$$

$$\forall P \quad s(-f,P)=-S(f,P)$$

$$\forall P \quad S(-f,P)=-s(f,P)$$

$$\boxed{\boxed{\int^{-}-f}}=\sup\{s(-f,P)\mid P\}=\sup\{-S(f,P)\mid P\}=$$

$$=-\inf\{S(f,P)\mid P\}=\boxed{-\int^{+}f}=-\int_a^b f(x)\,dx$$

$$\boxed{\int^{+}-f}=\ldots\ldots=-\int_a^b f(x)\,dx$$

$$\int_a^b -f(x)\,dx=-\int_a^b f(x)\,dx$$